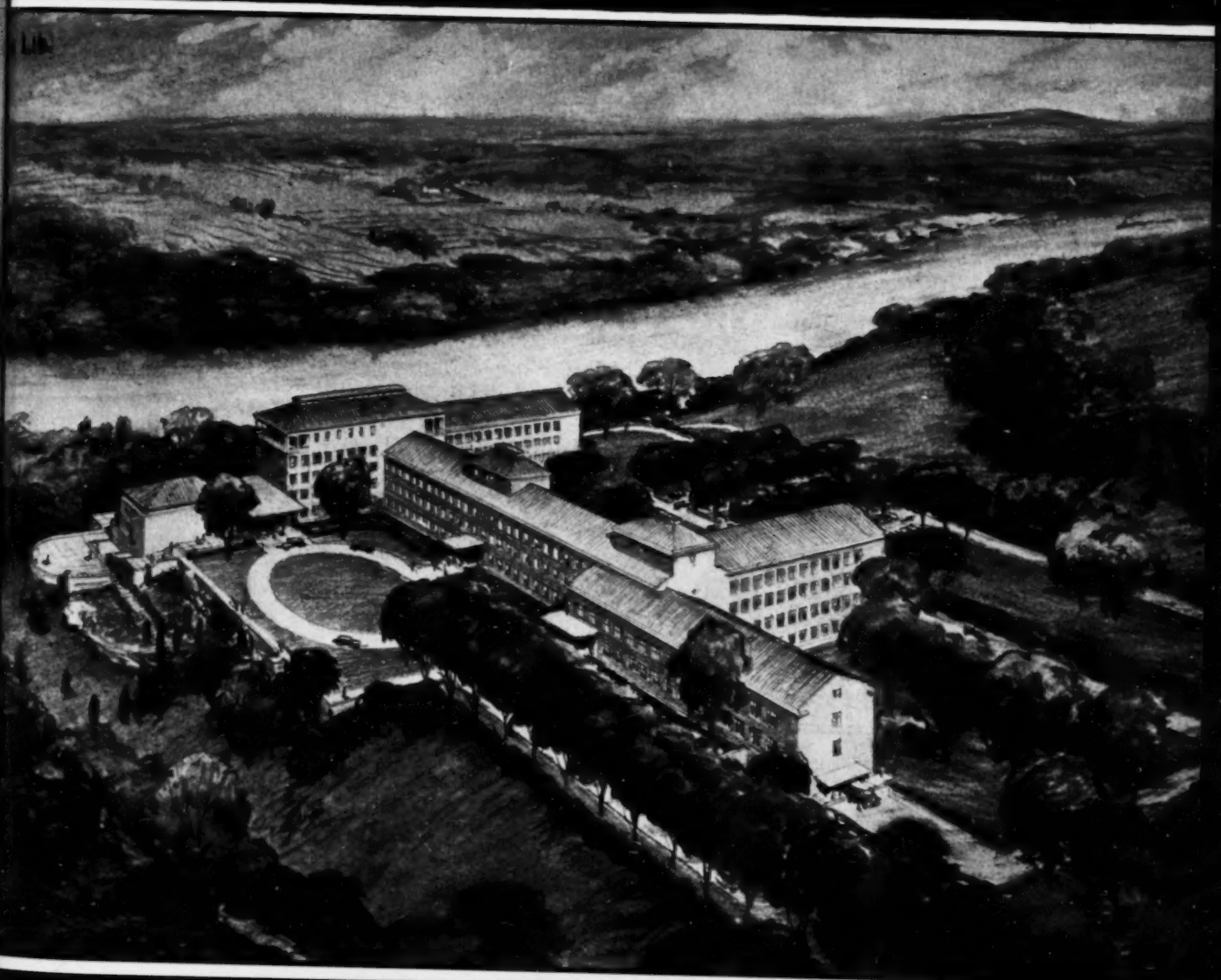
JOURNAL OF APPLIED PHYSICS



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News of the Hour

THE past few weeks have, indeed, been momentous ones for the public in general but also for the scientist in particular. On July 5 Dr. Vannevar Bush made his comprehensive report to the President on a program for post-war scientific research entitled "Science, the Endless Frontier."[1] On August 6 President Truman announced that a huge research and development program had led to a new bomb which makes use of the tremendous stored-up energy of atomic nuclei, and that this bomb had been used some 16 hours earlier against the enemy. On August 14 Japan capitulated and many war research secrets are being made public. First, a report on radar[2] was released by the Joint Board on Scientific Information Policy which described briefly how radar works and how it has been used in modern warfare. Next came the comprehensive and dramatic report[3] on the development of the methods of using atomic energy for military purposes, prepared by Dr. H. D. Smyth and released by the War Department. Many other reports are promised shortly by the Joint Board on Scientific Information Policy.

[1] Vannevar Bush, Science, the Endless Frontier. A Report to the President. July 1945. (184 pp.) U. S. Government Printing Office, Washington, 1945.

[2] Radar—A Report on Science at War. Released by the Joint Board on Scientific Information Policy for Office of Scientific Research and Development, War Department, and Navy Department. (53 pp.) For sale by the Superintendent of Documents, U. S. Government Printing Office, Washington 25, D. C. Price 15 cents.

[3] H. D. Smyth. A General Account of the Developments of Methods of Using Atomic Energy for Military Purposes under the Auspices of the U. S. Government, 1940–1945. Publication authorized by the War Department August 1945. (196 pp.) Expected to be made available through the U. S. Government Printing Office, The Princeton Press, and the REVIEWS OF MODERN PHYSICS.

The Editor of this JOURNAL would like to publish all of these reports in full, but before this can be done, they will be available from so many different sources that such a procedure is hardly justified, particularly in view of the large amount of original research work which will soon need to be published. It does seem desirable, however, to summarize these reports and print generous excerpts for the benefit of those readers who are unable to take the time to read the reports in full. These reports will be reviewed in the order mentioned above.

Science, the Endless Frontier[1]

In a letter dated November 17, 1944, President Roosevelt requested Dr. Bush's recommendations on the following points:

(1) What can be done, consistent with military security, and with the prior approval of the military authorities, to make known to the world as soon as possible the contributions which have been made during our war effort to scientific knowledge?

(2) With particular reference to the war of science against disease, what can be done now to organize a program for continuing in the future the work which has been done in medicine and related sciences?

(3) What can the Government do now and in the future to aid research activities by public and private organizations?

(4) Can an effective program be proposed for discovering and developing scientific talent in American youth so that the continuing future of scientific research in this country may be assured on a level comparable to what has been done during the war?

In seeking answers to President Roosevelt's questions, Dr. Bush had the assistance of several distinguished committees specially qualified to advise on these subjects. After these committees made their reports, Dr. Bush summarized the findings of the committees and made a single recommendation. His report contains as appendices the full reports of the various committees. Dr. Bush, in his summary of the report, emphasizes that scientific progress is essential for the war against diease, for our national security, and for the public welfare. He points out that we must renew our scientific talent, not only by discovering new and youthful talent, but also by utilizing as quickly as possible the potential scientific talent now in uniform.

In proposing a plan for action, Dr. Bush states:

"There are certain basic principles which must underlie the program of Government support for scientific research and education if such support is to be effective and if it is to avoid impairing the very things we seek to foster. These principles are as follows:

(1) Whatever the extent of support may be, there must be stability of funds over a period of years so that long-range programs may be undertaken.

(2) The agency to administer such funds should be composed of citizens selected only on the basis of their interest in and capacity to promote the work of the agency. They should be persons of broad interest in and understanding of the peculiarities of scientific research and education.

(3) The agency should promote research through contracts or grants to organizations outside the Federal Government. It should not operate any laboratories of its own.

(4) Support of basic research in the public and private colleges, universities, and research institutes must leave the internal control of policy, personnel, and the method and scope of the research to the institutions themselves. This is of the utmost importance.[4]

(5) While assuring complete independence and freedom for the nature, scope, and methodology of research carried on in the institutions receiving public funds, and while retaining discretion in the allocation of funds among such institutions, the Foundation proposed herein must be responsible to the President and the Congress. Only through such responsibility can we maintain the proper relationship between science and other aspects of a democratic system. The usual controls of audits, reports, budgeting, and the like should, of course, apply to the administrative and fiscal operations of the Foundation, subject, however, to such adjustments in procedure as are necessary to meet the special requirements of research.

"Basic research is a long-term process; it ceases to be basic if immediate results are expected on short-term support. Methods should therefore be found which will permit the agency to make commitments of funds from current appropriations for programs of five years' duration or longer. Continuity and stability of the program and its support may be expected (a) from the growing realization by the Congress of the benefits to the public from scientific research, and (b) from the conviction which will grow among those who conduct research under the auspices of the agency that good quality work will be followed by continuing support."

The specific recommendation of Dr. Bush's report is the creation of a National Research Foundation characterized as follows:

I. Purposes.—The National Research Foundation should develop and promote a national policy for scientific research and scientific education, should support basic research in nonprofit organizations, should develop scientific talent in American youth by means of scholarships and fellowships, and should by contract and otherwise support long-range research on military matters.

II. Members.—1. Responsibility to the people, through the President and the Congress, should be placed in the hands of, say, nine Members, who should be persons not otherwise connected with the Government and not representative of any special interest, who should be known as National Research Foundation Members, selected by the President on the basis of their interest in and capacity to promote the purposes of the Foundation.

2. The terms of the Members should be, say, 4 years, and no Member should be eligible for immediate reappointment provided he has served a full 4-year term. It should be arranged that the Members first appointed serve terms of such length that at least two Members are appointed each succeeding year.

3. The Members should serve without compensation

[4] This point has been the subject of some controversy. See New York Times, Editorial, July 21, 1945; Letter to the Editor by Dr. J. B. Conant, August 13, 1945; column, "Science in Review" by Dr. W. Kaempffert, August 19, 1945. Dr. Conant distinguishes between the methods in pure and applied science as follows: "There is only one proved method of assisting the advancement of pure science—that of picking men of genius, backing them heavily and leaving them to direct themselves. There is only one proved method of getting results in applied science—picking men of genius, backing them heavily, and keeping their aim on the target chosen."

but should be entitled to their expenses incurred in the performance of their duties.

4. The Members should elect their own chairman annually.

5. The chief executive officer of the Foundation should be a director appointed by the Members. Subject to the direction and supervision of the Foundation Members (acting as a board), the director should discharge all the fiscal, legal, and administrative functions of the Foundation. The director should receive a salary that is fully adequate to attract an outstanding man to the post.

6. There should be an administrative office responsible to the director to handle in one place the fiscal, legal, personnel, and other similar administrative functions necessary to the accomplishment of the purposes of the Foundation.

7. With the exception of the director, the division members, and one executive officer appointed by the director to administer the affairs of each division, all employees of the Foundation should be appointed under Civil Service regulations.

III. Organization.—1. In order to accomplish the purposes of the Foundation the Members should establish several professional Divisions to be responsible to the Members. At the outset these Divisions should be:

a. Division of Medical Research.—The function of this Division should be to support medical research.

b. Division of Natural Sciences.—The function of this Division should be to support research in the physical and natural sciences.

c. Division of National Defense.—It should be the function of this Division to support long-range scientific research on military matters.

d. Division of Scientific Personnel and Education.—It should be the function of this Division to support and to supervise the grant of scholarships and fellowships in science.[5]

e. Division of Publications and Scientific Collaboration.—This Division should be charged with encouraging the publication of scientific knowledge and promoting international exchange of scientific information.

2. Each Division of the Foundation should be made up of at least five members, appointed by the Members of the Foundation. In making such appointments the Members should request and consider recommendations from the National Academy of Sciences, which should be asked to establish a new National Research Foundation nominating committee in order to bring together the recommendations of scientists in all organizations. The chairman of each Division should be appointed by the Members of the Foundation.

3. The division Members should be appointed for such terms as the Members of the Foundation may determine, and may be reappointed at the discretion of the Members. They should receive their expenses and compensation for their services at a per diem rate of, say, $50 while engaged on business of the Foundation, but no division member should receive more than, say, $10,000 compensation per year.

4. Membership of the Division of National Defense should include, in addition to, say, five civilian members, one representative designated by the Secretary of War, and one representative of the Secretary of the Navy, who should serve without additional compensation for this duty.

IV. Functions.—1. The Members of the Foundation should have the following functions, powers, and duties:

a. To formulate over-all policies of the Foundation.

b. To establish and maintain such offices within the United States, its territories and possessions, as they may deem necessary.

c. To meet and function at any place within the United States, its territories and possessions.

d. To obtain and utilize the services of other Government agencies to the extent that such agencies are prepared to render such services.

e. To adopt, promulgate, amend, and rescind rules and regulations to carry out the provisions of the legislation and the policies and practices of the Foundation.

f. To review and balance the financial requirements of the several Divisions and to propose to the President the annual estimate for the funds required by each Division. Appropriations should be earmarked for the purposes of specific Divisions, but the Foundation should be left discretion with respect to the expenditure of each Division's funds.

g. To make contracts or grants for the conduct of research by negotiation without advertising for bids.

And with the advice of the National Research Foundation Divisions concerned—

h. To create such advisory and cooperating agencies and councils, state, regional, or national, as in their judgment will aid in effectuating the purposes of the legislation, and to pay the expenses thereof.

i. To enter into contracts with or make grants to educational and nonprofit research institutions for support of scientific research.

j. To initiate and finance in appropriate agencies, institutions, or organizations, research on problems related to the national defense.

k. To initiate and finance in appropriate organizations research projects for which existing facilities are unavailable or inadequate.

l. To establish scholarships and fellowships in the natural sciences including biology and medicine.

m. To promote the dissemination of scientific and technical information and to further its international exchange.

n. To support international cooperation in science by providing financial aid for international meetings, associations of scientific societies, and scientific research programs organized on an international basis.

o. To devise and promote the use of methods of improving the transition between research and its practical application in industry.

[5] Dr. K. Lark-Horovitz has commented in a private communication to the editor that there should be in addition support to insure the highest possible caliber of high school science teachers.

2. The Divisions should be responsible to the Members of the Foundation for—

a. Formulation of programs and policy within the scope of the particular Divisions.

b. Recommendations regarding the allocation of research programs among research organizations.

c. Recommendation of appropriate arrangements between the Foundation and the organizations selected to carry on the program.

d. Recommendation of arrangements with State and local authorities in regard to cooperation in a program of science scholarships and fellowships.

e. Periodic review of the quality of research being conducted under the auspices of the particular Division and revision of the program of support of research.

f. Presentation of budgets of financial needs for the work of the Division.

g. Maintaining liaison with other scientific research agencies, both governmental and private, concerned with the work of the Division.

V. Patent Policy.—The success of the National Research Foundation in promoting scientific research in this country will depend to a very large degree upon the cooperation of organizations outside the Government. In making contracts with or grants to such organizations the Foundation should protect the public interest adequately and at the same time leave the cooperating organization with adequate freedom and incentive to conduct scientific research. The public interest will normally be adequately protected if the Government receives a royalty-free license for governmental purposes under any patents resulting from work financed by the Foundation. There should be no obligation on the research institution to patent discoveries made as a result of support from the Foundation. There should certainly *not* be any absolute requirement that all rights in such discoveries be assigned to the Government, but it should be left to the discretion of the director and the interested Division whether in special cases the public interest requires such an assignment. Legislation on this point should leave to the Members of the Foundation discretion as to its patent policy in order that patent arrangements may be adjusted as circumstances and the public interest require.

VI. Special Authority.—In order to insure that men of great competence and experience may be designated as Members of the Foundation and as members of the several professional Divisions, the legislation creating the Foundation should contain specific authorization so that the Members of the Foundation and the Members of the Divisions may also engage in private and gainful employment, notwithstanding the provisions of any other laws: provided, however, that no compensation for such employment is received in any form from any profit-making institution which receives funds under contract, or otherwise, from the Division or Divisions of the Foundation with which the individual is concerned. In normal times, in view of the restrictive statutory prohibitions against dual interests on the part of Government officials, it would be virtually impossible to persuade persons having private employment of any kind to serve the Government in an official capacity. In order, however, to secure the part-time services of the most competent men as Members of the Foundation and the Divisions, these stringent prohibitions should be relaxed to the extent indicated.

Since research is unlike the procurement of standardized items, which are susceptible to competitive bidding on fixed specifications, the legislation creating the National Research Foundation should free the Foundation from the obligation to place its contracts for research through advertising for bids. This is particularly so since the measure of a successful research contract lies not in the dollar cost but in the qualitative and quantitative contribution which is made to our knowledge. The extent of this contribution in turn depends on the creative spirit and talent which can be brought to bear within a research laboratory. The National Research Foundation must, therefore, be free to place its research contracts or grants not only with those institutions which have a demonstrated research capacity but also with other institutions whose latent talent or creative atmosphere affords promise of research success.

As in the case of the research sponsored during the war by the Office of Scientific Research and Development, the research sponsored by the National Research Foundation should be conducted, in general, on an actual cost basis without profit to the institution receiving the research contract or grant.

There is one other matter which requires special mention. Since research does not fall within the category of normal commercial or procurement operations which are easily covered by the usual contractual relations, it is essential that certain statutory and regulatory fiscal requirements be waived in the case of research contractors. For example, the National Research Foundation should be authorized by legislation to make, modify, or amend contracts of all kinds with or without legal consideration, and without performance bonds. Similarly, advance payments should be allowed in the discretion of the Director of the Foundation when required. Finally, the normal vouchering requirements of the General Accounting Office with respect to detailed itemization or substantiation of vouchers submitted under cost contracts should be relaxed for research contractors. Adherence to the usual procedures in the case of research contracts will impair the efficiency of research operations and will needlessly increase the cost of the work to the Government. Without the broad authority along these lines which was contained in the First War Powers Act and its implementing Executive Orders, together with the special relaxation of vouchering requirements granted by the General Accounting Office, the Office of Scientific Research and Development would have been gravely handicapped in carrying on research on military matters during the war. Colleges and universities in which research will be conducted principally under contract with the Foundation are, unlike commercial institutions, not equipped to handle the detailed vouchering procedures and auditing technicalities which are required of the usual Government contractors.

VII. Budget.—Studies by the several committees provide a partial basis for making an estimate of the order of magnitude of the funds required to implement the proposed program. Clearly the program should grow in a healthy manner from modest beginnings. The following very rough estimates are given for the first year of operation after the Foundation is organized and operating, and for the fifth year of operation, when it is expected that the operations would have reached a fairly stable level:

Acitivity	*Millions of dollars* *First year*	*Fifth year*
Division of Medical Research	$ 5.0	$ 20.0
Division of Natural Sciences	10.0	50.0
Division of National Defense	10.0	20.0
Division of Scientific Personnel and Education	7.0	29.0
Division of Publications and Scientific Collaboration	.5	1.0
Administration	1.0	2.5
	$33.5	$122.5

. . . . . .

Dr. Bush then speaks of definite action by Congress. "The National Research Foundation herein proposed meets the urgent need of the days ahead. The form of the organization suggested is the result of considerable deliberation. The form is important. The very successful pattern of organization of the National Advisory Committee for Aeronautics, which has promoted basic research on problems of flight during the past thirty years, has been carefully considered in proposing the method of appointment of Members of the Foundation and in defining their responsibilities. Moreover, whatever program is established, it is vitally important that it satisfy the Five Fundamentals.

"Legislation is necessary. It should be drafted with great care.[6] Early action is imperative, however, if this nation is to meet the challenge of science and fully utilize the potentialities of science. On the wisdom with which we bring science to bear against the problems of the coming years depends in large measure our future as a nation."

Radar—A Report on Science at War[2]

Radar comes from the descriptive phrase, "radio detection and ranging." It would be more appropriate, perhaps, to make the phrase "radio direction-finding and ranging," since the direction and the range of objects in its field of view are the two basic qualities radar has to offer. The significant fact about radar is that it can see farther than the eye can, even in the best visibility; and radar's ability to see is relatively unaffected by night, fog, smoke, or rain.

"Radar, in consequence, has played a great and increasing role right from the beginning of the present war. It has, more than any single development since the airplane, changed the face of warfare; for one of the greatest weapons in any war is surprise, and surprise is usually achieved by concealment in the last minutes or hours before an attack. The concealment formerly afforded by darkness or fog or cloud or artificial smoke or the glare of the sun simply does not exist in the world of radar."

"It will be clear that the word radar refers to no single instrument. Indeed, an individual radar set may be a hundred-pound outfit, the size and shape of a small bomb, for installation in a fast airplane; or it may be a sprawling complex of shacks and trucks with its own telephone central, with a giant antenna structure, and with a whole company of soldiers to man it. It may be 5 tons of equipment disposed here and there as in a fast carrier, or it may be a couple of water-tight boxes on deck and a modest bulge on the mast of a PT boat. Despite the great physical diversity of the forms in which radar is used and embodied, a common set of principles is back of every radar, and once these principles are recognized radar is understood."

"*Technical Description of Radar Systems.* Practically every radar set is made up of the following major part or components: (1) a modulator; (2) a radio-frequency oscillator; (3) an antenna with suitable scanning mechanism; (4) a receiver; and (5) an indicator.

"While the physical form of each of these components may vary widely from one kind of radar set to another, each radar must have this complement of parts in order to function.

"1. The modulator is a device for taking power from the primary power source (which may be the commercial power line, a special engine or motor-driven generator, or storage batteries) and forming suitable voltage pulses to drive the r-f oscillator in its bursts of radio-frequency oscillations. In other words, it is the modulator which turns on the radio-frequency oscillator to oscillate violently for a millionth of a second or so, turns it off sharply and keeps it in repose until time for the next burst.

"2. The radio-frequency oscillator is a vacuum tube of suitable design, or a group of such tubes, which will oscillate at the desired radio frequency and give the desired

[6] Two different bills have already been introduced into the Congress. One is by Senator H. M. Kilgore from West Virginia and the other by Senator W. G. Magnuson from Washington. Hearings on these bills will begin shortly.

bursts of radio-frequency power when connected to the modulator. The development of suitable oscillator tubes has been one of the major achievements of the radar art. It is a relatively simple job to produce a radio-frequency oscillator which will give oscillations of any desired frequency provided one is satisfied with a power of only a few thousandths of a watt. In the receiving part of a radar circuit this amount of power is adequate. A practical radar transmitter, however, must generate during its momentary bursts of oscillation a power which may run into hundreds of kilowatts. Since the oscillator is turned on a small fraction of the time, the average power is usually hundreds of times less than the peak power, but even the average power may run up to the order of one kilowatt. Thus, practical radar equipment requires extremely high frequency oscillators running at powers thousands of times greater than was thought possible a few years ago.

"3. The problem of antenna design is also one of the major problems in radar, incomprehensible as this may seem to the operator of a home radio receiver, who finds a few yards of wire strung up on his roof adequate for his purpose. A suitable radar antenna must have the following characteristics:

"a. It must be directional; that is, it must concentrate the radio energy into a definitely defined beam, since this is the method by which the direction to the objects detected is determined;

"b. It must be highly efficient. All of the generated power must go into the beam and none must leak off into 'side lobes' in other directions, since such side lobes may often be fatally confusing; and,

"c. The radar antenna must be capable of being directed or scanned from one point in space to another, and on shipboard and in aircraft it must frequently be stabilized to take out the motions of the ship or airplane itself.

"An antenna may be made directional either by building it up of an array of small antennas or dipoles, suitably spaced and phased to concentrate the energy in one direction, or it may be built on the searchlight principle of spraying the energy into a large parabolic 'mirror,' which focuses the energy into a beam. In either case, the larger the antenna, the sharper the beam for any given wavelength. Sometimes antennas may be longer in one direction than the other, giving a beam which is sharper in the first direction and thus fan shaped.

"The scanning of the portion of space which the radar set is intended to cover must usually be done by mechanical movement of the antenna structure itself. This means that the structure, whatever its size, must swing around or up and down to direct the beam in the necessary direction. In certain cases where one needs to scan only a small sector, techniques have been worked out for rapid electrical scanning not requiring the motion of the whole antenna structure itself. So far, however, there has been no method for extending this rapid electrical scanning to cover more than a relatively small sector. Radars for directing guns which need accurate and fast data in a small sector are making use, however, of this valuable technique.

"To carry the radio-frequency energy from the oscillator to the antenna, and the echo from the antenna to the receiver, wires and coaxial cables are used at ordinary wave-lengths. For microwaves, however, it is more efficient to use wave guides, which essentially are carefully proportioned hollow pipes—and the transmission system hence is often called 'plumbing.'

"4. The problem of the receiver for radar is also a complex one. In practically all radars the superheterodyne principle is employed, which involves generating at low power a radio frequency fairly close to that received, and "beating" this against the received signals, forming an intermediate frequency, which is then amplified many times. Curiously enough the crystal, used as a detector and mixer, has again come into its own in microwave receivers. The peculiar characteristics of pulse signals require that receivers be built with extremely fast response, much faster even than that required in television. The final stages must prepare the signals for suitable presentation in the indicator. The receiver normally occupies a relatively small box in the complete radar set, and yet this box represents a marvel of engineering ingenuity. A particularly difficult piece of development is concerned with a part closely connected with the receiver. This is a method of disconnecting the receiver from the antenna during intervals when the transmitter is operating so that the receiver will not be paralyzed or burned out by the stupendous bursts of radio-frequency energy generated by the transmitter. Within a millionth of a second after the transmitter has completed its pulse, however, the receiver must be open to receive the relatively weak echo signals; but now the transmitter part of the circuit must be closed off so it will not absorb any of this energy.

"5. It is the indicator of a radar that presents the information collected in a form best adapted to efficient use of the set. Nearly (but not quite) all radar indicators consist of one or more cathode-ray tubes. In the simplest or 'A' type of presentation the electron beam is given a deflection proportional to time in one direction—say, horizontally—and proportional to the strength of the echo pulse in the other—say, vertically. If no signals are visible, then one sees a bright horizontal line (the 'time base') across the tube face, the distance along this line representing time elapsed after the outgoing pulse. A returning echo then gives a V-shaped break in the line at the point corresponding to the time it took the echo to come back. The position of the 'pip' along this line measures the distance to the reflecting object. There are many variations of this type of indicator for special purposes, but most radars have an A-scope, even when other types are also provided.

"Many types of radar whose antennas 'scan' various directions employ the PPI tube. Here the time base starts from the center of the tube and moves radially outward in a direction corresponding to that in which the antenna is pointing. This time base rotates in synchronism with the antenna. The returning signal, instead of causing a break in the time base, simply intensifies its brilliance for an instant. Hence each signal appears as a bright spot of

light at a position corresponding to the range and bearing of the target. Thus a maplike picture of all reflecting objects appears in the cathode-ray tube face.

"Since the antenna can usually be rotated only slowly (e.g., from 1 to 20 r.p.m.) and since the light from an ordinary cathode-ray tube fades away almost instantly, one might expect not to see a 'map' at all, but only bright flashes at various spots as the antenna revolves. Some way had to be found to make the brightness of these flashes persist for many seconds after they were produced. Special screens were developed which continue to glow for some time after being lighted by a signal. Thus the whole map is displayed at once."

"*Radar in the Peacetime World.* There has already been a great deal of rather uninformed speculation about the peaceful uses of radar. It will be clear from what has gone before that the direct and immediate use of radar will be to make air and sea navigation entirely continuous and foolproof, regardless of night or weather. Its use in land transport, during the immediate postwar years, is more dubious. In the forms in which it exists now, radar is not a very useful attachment to an automobile or a railroad locomotive.

"The biggest influence radar will have after the war is indirect. The thousands of man-years which have gone into the improvement of the detailed components which make up a radar set —many of these components being identical with those of a radio or television set, or hearing aid, or other electronic device—have made obsolete many of our prewar ideas about what could and could not be done in electronics.

"Furthermore, radar has made the electronic industry one of America's major ones, now comparable in size to the prewar automobile industry. This new industry, through its enormous laboratories, can be expected to find innumerable applications in a wide variety of fields.

"If television is still around the corner after the war, nothing but economic factors, not technical ones, will have kept it there. Communication, especially radio communication, will have a tremendous flowering based largely on the opening up of the microwave field.

"Individual radio communication is even beginning to appear a practical matter, subject to certain limitations.

"The number of men who have been trained in the techniques of radar operation and maintenance by the Army and the Navy is colossal; we can expect these men, in large part, to make feasible the greatly expanded use of electronic equipment of all kinds, because of their preparation to enter the industry or to set up in the parts and repair business.

"Altogether, it is fair to say that radar, as radar, will have a mild immediate beneficial effect on all our lives, by making it safer to travel by sea or by air. But the impact on electronics generally of techniques developed during the war because of radar will have profound and far-reaching effects on the shape of our daily life."

Atomic Energy for Military Purposes[3]

"The purpose of this report is to describe the scientific and technical developments in this country since 1940 directed toward the military use of energy from atomic nuclei. Although not written as a 'popular' account of the subject, this report is intended to be intelligible to scientists and engineers generally and to other college graduates with a good grounding in physics and chemistry. The equivalence of mass and energy is chosen as the guiding principle in the presentation of the background material of the 'Introduction.'"

"One conclusion that appeared rather early in the development of the theory of relativity was that the inertial mass of a moving body increased as its speed increased. This implied an equivalence between an increase in energy of motion of a body, that is, its kinetic energy, and an increase in its mass. To most practical physicists and engineers this appeared a mathematical fiction of no practical importance. Even Einstein could hardly have foreseen the present applications, but as early as 1905 he did clearly state that mass and energy were equivalent and suggested that proof of this equivalence might be found by the study of radioactive substances. He concluded that the amount of energy, E, equivalent to a mass, m, was given by the equation

$$E = mc^2$$

where c is the velocity of light. If this is stated in actual numbers, its startling character is apparent. It shows that one kilogram (2.2 pounds) of matter, if converted entirely into energy, would give 25 billion kilowatt hours of energy. This is equal to the energy that would be generated by the total electric power industry in the United States (as of 1939) running for approximately two months. Compare this fantastic figure with the 8.5 kilowatt hours of heat energy which

may be produced by burning an equal amount of coal."

In this report Professor Smyth proceeds then with an excellent summary of our present knowledge in nuclear physics. He speaks about the relation of atomic number to the charge on the nucleus, the mass number and its relation to atomic weight, the nature of alpha-, beta-, and gamma-rays, natural and artificial radioactivity, and about the neutron, the positron, and the deuteron and their appearance in nuclear reactions. He points out that the difference between the mass of an atom itself and the mass number is a measure of the binding energy of the nucleus. For instance, from this difference one calculates that the energy required to break up all of the helium atoms in a gram of helium is 1.62 times 10^{11} gram calories, or 190,000 kilowatt hours.

Professor Smyth continues his discussion with a statement of the practicability of atomic power in 1939. "We have talked glibly about the equivalence of mass and energy and about nuclear reactions, such as that of protons on lithium, where energy was released in relatively large amounts. Now let us ask why atomic power plants did not spring up all over the world in the thirties. After all, if we can get 2.76×10^{-5} erg from an atom of lithium struck by a proton, we might expect to obtain approximately half a million kilowatt hours by combining a gram of hydrogen with seven grams of lithium. It looks better than burning coal. The difficulties are in producing the high speed protons and in controlling the energy produced. All the experiments we have been talking about have been done with very small quantities of material, large enough in numbers of atoms, to be sure, but in terms of ordinary masses infinitesimal—not tons or pounds or grams, but fractions of micrograms. The amount of energy used up in the experiment was always far greater than the amount generated by the nuclear reaction.

"Neutrons are particularly effective in producing nuclear disintegration. Why weren't they used? If their initial source was an ion beam striking a target, the limitations discussed in the last paragraph applied. If a radium and beryllium source was to be used, the scarcity of radium was a difficulty.

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"Our common sources of power, other than sunlight and water power, are chemical reactions —usually the combustion of coal or oil. They release energy as the result of rearrangements of the outer electronic structures of the atoms, the same kind of process that supplies energy to our bodies. Combustion is always self-propagating; thus lighting a fire with a match releases enough heat to ignite the neighboring fuel, which releases more heat which ignites more fuel, and so on. In the nuclear reactions we have described this is not generally true; neither the energy released nor the new particles formed are sufficient to maintain the reaction. But we can imagine nuclear reactions emitting particles of the same sort that initiate them and in sufficient numbers to propagate the reaction in neighboring nuclei. Such a self-propagating reaction is called a 'chain reaction' and such conditions must be achieved if the energy of the nuclear reactions with which we are concerned is to be put into large-scale use."

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Professor Smyth's story of the discovery of nuclear fission is especially interesting. "As has already been mentioned, the neutron proved to be the most effective particle for inducing nuclear changes. This was particularly true for the elements of highest atomic number and weight where the large nuclear charge exerts strong repulsive forces on deuteron or proton projectiles but not on uncharged neutrons. The results of the bombardment of uranium by neutrons had proved interesting and puzzling. First studied by Fermi and his colleagues in 1934, they were not properly interpreted until several years later.

"On January 16, 1939, Niels Bohr of Copenhagen, Denmark, arrived in this country to spend several months in Princeton, N. J., and was particularly anxious to discuss some abstract problems with A. Einstein. (Four years later Bohr was to escape from Nazi-occupied Denmark in a small boat.) Just before Bohr left Denmark two of his colleagues, O. R. Frisch and L. Meitner (both refugees from Germany), had told him their guess that the absorption of a neutron by a uranium nucleus sometimes caused that nucleus to split into approximately equal parts with the release of enormous quantities of energy, a process that soon began to be called nuclear 'fission.' The occasion for this hypothesis was the important discovery of O. Hahn and F. Strassmann in Germany (published in *Naturwissenschaften* in early January 1939) which proved that an isotope of barium was

produced by neutron bombardment of uranium. Immediately on arrival in the United States Bohr communicated this idea to his former student J. A. Wheeler and others at Princeton, and from them the news spread by word of mouth to neighboring physicists including E. Fermi at Columbia University. As a result of conversations between Fermi, J. R. Dunning, and G. B. Pegram, a search was undertaken at Columbia for the heavy pulses of ionization that would be expected from the flying fragments of the uranium nucleus. On January 26, 1939, there was a Conference on Theoretical Physics at Washington, D. C., sponsored jointly by the George Washington University and the Carnegie Institution of Washington. Fermi left New York to attend this meeting before the Columbia fission experiments had been tried. At the meeting Bohr and Fermi discussed the problem of fission, and in particular Fermi mentioned the possibility that neutrons might be emitted during the process. Although this was only a guess, its implication of the possibility of a chain reaction was obvious. A numer of sensational articles were published in the press on this subject. Before the meeting in Washington was over, several other experiments to confirm fission had been initiated, and positive experimental confirmation was reported from four laboratories (Columbia University, Carnegie Institution of Washington, Johns Hopkins University, University of California) in the February 15, 1939, issue of *The Physical Review*. By this time Bohr had heard that similar experiments had been made in his laboratory in Copenhagen about January 15th. (Letter by Frisch to *Nature* dated January 16, 1939 and appearing in the February 18th issue.) F. Joliot in Paris had also published his first results in the *Comptes Rendus* of January 30, 1939. From this time on there was a steady flow of papers on the subject of fission, so that by the time (December 6, 1939) Turner wrote a review article on the subject in the *Reviews of Modern Physics* nearly one hundred papers had appeared. Complete analysis and discussion of these papers have appeared in Turner's article and elsewhere."

Professor Smyth then summarizes the state of nuclear physics in 1940. "By 1940 nuclear reactions had been intensively studied for over ten years. Several books and review articles on nuclear physics had been published. New techniques had been developed for producing and controlling nuclear projectiles, for studying artificial radioactivity, and for separating submicroscopic quantities of chemical elements produced by nuclear reactions. Isotope masses had been measured accurately. Neutron-capture cross sections had been measured. Methods of slowing down neutrons had been developed. Physiological effects of neutrons had been observed; they had even been tried in the treatment of cancer. All such information was generally available; but it was very incomplete. There were many gaps and many inaccuracies. The techniques were difficult and the quantities of materials available were often sub-microscopic. Although the fundamental principles were clear, the theory was full of unverified assumptions and calculations were hard to make. Predictions made in 1940 by different physicists of equally high ability were often at variance. The subject was in all-too-many respects an art, rather than a science.

"Looking back on the year 1940, we see that all the prerequisites to a serious attack on the problem of producing atomic bombs and controlling atomic power were at hand. It had been proved that mass and energy were equivalent. It had been proved that the neutrons initiating fission of uranium reproduced themselves in the process and that therefore a multiplying chain reaction might occur with explosive force. To be sure, no one knew whether the required conditions could be achieved, but many scientists had clear ideas as to the problems involved and the directions in which solutions might be sought."

Professor Smyth then comes to the atomic bomb itself. "The principle of operation of an atomic bomb or power plant utilizing uranium fission is simple enough. If one neutron causes a fission that produces more than one new neutron, the number of fissions may increase tremendously with the release of enormous amounts of energy. It is a question of probabilities. Neutrons produced in the fission process may escape entirely from the uranium, may be captured by uranium in a process not resulting in fission, or may be captured by an impurity. Thus the question of whether a chain reaction does or does not go depends on the result of a competition among four processes:

(1) escape,
(2) non-fission capture by uranium,
(3) non-fission capture by impurities,
(4) fission capture.

If the loss of neutrons by the first three processes

is less than the surplus produced by the fourth, the chain reaction occurs; otherwise it does not. Evidently any one of the first three processes may have such a high probability in a given arrangement that the extra neutrons created by fission will be insufficient to keep the reaction going. For example, should it turn out that process (2)—non-fission capture by uranium—has a much higher probability than fission capture, there would presumably be no possibility of achieving a chain reaction."

"The relative number of neutrons which escape from a quantity of uranium can be minimized by changing the size and shape. In a sphere any surface effect is proportional to the square of the radius, and any volume effect is proportional to the cube of the radius. Now the escape of neutrons from a quantity of uranium is a surface effect depending on the area of the surface, but fission capture occurs throughout the material and is therefore a volume effect. Consequently the greater the amount of uranium, the less probable it is that neutron escape will predominate over fission capture and prevent a chain reaction. Loss of neutrons by non-fission capture is a volume effect like neutron production by fission capture, so that increase in size makes no change in its relative importance.

"The critical size of a device containing uranium is defined as the size for which the production of free neutrons by fission is just equal to their loss by escape and by non-fission capture. In other words, if the size is smaller than critical, then—by definition—no chain reaction will sustain itself. In principle it was possible in 1940 to calculate the critical size, but in practice the uncertainty of the constants involved was so great that the various estimates differed widely. It seemed not improbable that the critical size might be too large for practical purposes. Even now estimates for untried arrangements vary somewhat from time to time as new information becomes available.

"In Chapter I we said that thermal neutrons have the highest probability of producing fission of U-235 but we also said that the neutrons emitted in the process of fission had high speeds. Evidently it was an oversimplification to say that the chain reaction might maintain itself if more neutrons were created by fission than were absorbed. For the probability both of fission capture and of non-fission capture depends on the speed of the neutrons. Unfortunately, the speed at which non-fission capture is most probable is intermediate between the average speed of neutrons emitted in the fission process and the speed at which fission capture is most probable.

"For some years before the discovery of fission, the customary way of slowing down neutrons was to cause them to pass through material of low atomic weight, such as hydrogenous material. It was E. Fermi and L. Szilard who proposed the use of graphite as a moderator for a chain reaction. The process of slowing down or moderation is simply one of elastic collisions between high-speed particles and particles practically at rest. The more nearly identical the masses of neutron and struck particle the greater the loss of kinetic energy by the neutron. Therefore the light elements are most effective as 'moderators,' i.e., slowing down agents, for neutrons.

"It occurred to a number of physicists that it might be possible to mix uranium with a moderator in such a way that the high-speed fission neutrons, after being ejected from uranium and before re-encountering uranium nuclei, would have their speeds reduced below the speeds for which non-fission capture is highly probable. Evidently the characteristics of a good moderator are that it should be of low atomic weight and that it should have little or no tendency to absorb neutrons. Lithium and boron are excluded on the latter count. Helium is difficult to use because it is a gas and forms no compounds. The choice of moderator therefore lay between hydrogen, deuterium, beryllium, and carbon. Even now no one of these substances can be excluded from the list of practical possibilities.

"The general scheme of using a moderator mixed with the uranium was pretty obvious. A specific manner of using a moderator was first suggested in this country, so far as we can discover, by Fermi and Szilard. The idea was to use lumps of uranium of considerable size imbedded in a matrix of moderator material. Such a lattice can be shown to have real advantages over a homogeneous mixture. As the constants were more accurately determined, it became possible to calculate theoretically the type of lattice that would be most effective."

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"So far, all our discussion has been primarily concerned with the use of uranium itself. We have already mentioned the suggestion that the element of atomic number 94 and mass 239,

commonly referred to as plutonium, might be very effective. Actually, we now believe it to be of value comparable to pure U-235. We have mentioned the difficulty of separating U-235 from the more abundant isotope U-238. These two isotopes are, of course, chemically identical. But plutonium, although produced from U-238, is a different chemical element. Therefore, if a process could be worked out for converting some of the U-238 to plutonium, a *chemical* separation of the plutonium from uranium might prove more practicable than the *isotopic* separation of U-235 from U-238.

"Suppose that we have set up a controllable chain reaction in a lattice of natural uranium and a moderator—say carbon, in the form of graphite. Then as the chain reaction proceeds, neutrons are emitted in the process of fission of the U-235 and many of these neutrons are absorbed by U-238. This produces U-239, each atom of which then emits a beta particle, becoming neptunium ($_{93}Np^{239}$). Neptunium, in turn, emits another beta particle, becoming plutonium ($_{94}Pu^{239}$), which emits an alpha particle, decaying again to U-235, but so slowly that in effect it is a stable element. If, after the reaction has been allowed to proceed for a considerable time, the mixture of metals is removed, it may be possible to extract the plutonium by chemical methods and purify it for use in a subsequent fission chain reaction of an explosive nature."

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"By the fall of 1942 enough graphite, uranium oxide, and uranium metal were available at Chicago to justify an attempt to build an actual self-sustaining chain-reacting pile. But the amount of metal available was small—only about 6 tons—and other materials were none too plentiful and of varying quality. These conditions rather than optimum efficiency controlled the design.

"The pile was constructed on the lattice principle with graphite as a moderator and lumps of metal or oxide as the reacting units regularly spaced through the graphite to form the lattice. Instruments situated at various points in the pile or near it indicated the neutron intensity, and movable strips of absorbing material served as controls. Since there were bound to be some neutrons present from spontaneous fission or other sources, it was anticipated that the reaction would start as soon as the structure had reached critical size if the control strips were not set in 'retard' position. Consequently, the control strips were placed in a suitable 'retard' position from the start and the neutron intensity was measured frequently. This was fortunate since the approach to critical condition was found to occur at an earlier stage of assembly than had been anticipated.

"The pile was first operated as a self-sustaining system on December 2, 1942. So far as we know, this was the first time that human beings ever initiated a self-maintaining nuclear chain reaction. Initially the pile was operated at a power level of ½ watt, but on December 12th the power level was raised to 200 watts."

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"The immediate object of building a uranium-graphite pile was to prove that there were conditions under which a chain reaction would occur, but the ultimate objective of the laboratory was to produce plutonium by a chain reaction. Therefore we are interested in the relation between the power at which a pile operates and the rate at which it produces plutonium. The relation may be evaluated to a first approximation rather easily. A pile running stably must be producing as many neutrons as it is losing. For every thermal neutron absorbed in U-235 a certain number of neutrons, η, is emitted. One of these neutrons is required to maintain the chain. Therefore, assuming the extra neutrons all are absorbed by U-238 to form plutonium, there will be $\eta-1$ atoms of Pu^{239} formed for every fission. Every fission releases roughly 200 Mev of energy. Therefore the formation of $\eta-1$ atoms of plutonium accompanies the release of about 200 Mev. Since $\eta-1$ is a small number, we can guess that to produce a kilogram a day of plutonium a chain-reacting pile must be releasing energy at the rate of 500,000 to 1,500,000 kilowatts. The first chain-reacting pile that we have just described operated at a maximum of 200 watts. Assuming that a single bomb will require the order of one to 100 kilograms of plutonium, the pile that has been described would have to be kept going at least 70,000 years to produce a single bomb."

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"On the basis of the evidence available it was clear that a plutonium production rate somewhere between a kilogram a month and a kilogram a day would be required. At the rate of a kilogram a day, a 500,000 to 1,500,000 kilowatt

plant would be required. (The ultimate capacity of the hydroelectric power plants at the Grand Coulee Dam is expected to be 2,000,000 kw.) Evidently the creation of a plutonium production plant of the required size was to be a major enterprise even without attempting to utilize the thermal energy liberated. Nevertheless, by November 1942 most of the problems had been well defined and tentative solutions had been proposed. Although these problems will be discussed in some detail in the next chapter, we will mention them here.

"Since a large amount of heat is generated in any pile producing appreciable amounts of plutonium, the first problem of design is a cooling system. Before such a system can be designed, it is necessary to find the maximum temperature at which a pile can run safely and the factors—nuclear or structural—which determine this temperature. Another major problem is the method for loading and unloading the uranium, a problem complicated by the shielding and the cooling system. Shielding against radiation has to be planned for both the pile itself and the chemical separation plant. The nature of the separation plant depends on the particular separation process to be used, which has to be decided. Finally, speed of procurement and construction must be primary factors in the planning of both the pile and the chemical plant."

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"It is beyond the scope of this report to give any account of the construction of the Hanford Engineer Works, but it is to be hoped that the full story of this extraordinary enterprise and the companion one, the Clinton Engineer Works, will be published at some time in the future. The Hanford site was examined by representatives of General Groves and of du Pont at the end of 1942, and use of the site was approved by General Groves after he had inspected it personally. It was on the west side of the Columbia River in central Washington north of Pasco. In the early months of 1943 a two-hundred square mile tract in this region was acquired by the government (by lease or purchase) through the Real Estate Division of the Office of the Chief of Engineers. Eventually an area of nearly a thousand square miles was brought under government control. At the time of acquisition of the land there were a few farms and two small villages, Hanford and Richland, on the site, which was otherwise sage-brush plains and barren hills. On the 6th of April, 1943, ground was broken for the Hanford construction camp. At the peak of activity in 1944, this camp was a city of 60,000 inhabitants, the fourth largest city in the state. Now, however, the camp is practically deserted as the operating crew is housed at Richland.

"Work was begun on the first of the Hanford production piles on June 7, 1943, and operation of the first pile began in September 1944. The site was originally laid out for five piles, but the construction of only three has been undertaken. Besides the piles, there are, of course, plutonium separation plants, pumping stations and water-treatment plants. There is also a low power chain-reacting pile for material testing. Not only are the piles themselves widely spaced for safety —several miles apart—but the separation plants are well away from the piles and from each other. All three piles were in operation by the summer of 1945."

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Professor Smyth then summarizes as follows:

"*Present Over-all Status.* As a result of the labors of the Manhattan District organization in Washington and in Tennessee, of the scientific groups at Berkeley, Chicago, Columbia, Los Alamos, and elsewhere, of the industrial groups at Clinton, Hanford, and many other places, the end of June 1945 finds us expecting from day to day to hear of the explosion of the first atomic bomb devised by man. All the problems are believed to have been solved at least well enough to make a bomb practicable. A sustianed neutron chain reaction resulting from nuclear fission has been demonstrated; the conditions necessary to cause such a reaction to occur explosively have been established and can be achieved; production plants of several different types are in operation, building up a stock pile of the explosive material. Although we do not know when the first explosion will occur nor how effective it will be, announcement of its occurrence will precede the publication of this report. Even if the first attempt is relatively ineffective, there is little doubt that later efforts will be highly effective; the devastation from a single bomb is expected to be comparable to that of a major air raid by usual methods.

"A weapon has been developed that is potentially destructive beyond the wildest nightmares of the imagination; a weapon so ideally suited to sudden unannounced attack that a country's

major cities might be destroyed overnight by an ostensibly friendly power. This weapon has been created not by the devilish inspiration of some warped genius but by the arduous labor of thousands of normal men and women working for the safety of their country. Many of the principles that have been used were well known to the international scientific world in 1940. To develop the necessary industrial processes from these principles has been costly in time, effort, and money, but the processes which we selected for serious effort have worked and several that we have not chosen could probably be made to work. We have an initial advantage in time because, so far as we know, other countries have not been able to carry out parallel developments during the war period. We also have a general advantage in scientific and particularly in industrial strength, but such an advantage can easily be thrown away.

"Before the surrender of Germany there was always a chance that German scientists and engineers might be developing atomic bombs which would be sufficiently effective to alter the course of the war. There was therefore no choice but to work on them in this country. Initially many scientists could and did hope that some principle would emerge which would prove that atomic bombs were inherently impossible. This hope has faded gradually; fortunately in the same period the magnitude of the necessary industrial effort has been demonstrated so that the fear of German success weakened even before the end came. By the same token, most of us are certain that the Japanese cannot develop and use this weapon effectively.

"*Prognostication.* As to the future, one may guess that technical developments will take place along two lines. From the military point of view it is reasonably certain that there will be improvements both in the processes of producing fissionable material and in its use. It is conceivable that totally different methods may be discovered for converting matter into energy since it is to be remembered that the energy released in uranium fission corresponds to the utilization of only about one-tenth of one percent of its mass. Should a scheme be devised for converting to energy even as much as a few percent of the matter of some common material, civilization would have the means to commit suicide at will.

"The possible uses of nuclear energy are not all destructive, and the second direction in which technical development can be expected is along the paths of peace. In the fall of 1944 General Groves appointed a committee to look into these possibilities as well as those of military significance. This committee (Dr. R. C. Tolman, chairman; Rear Admiral E. W. Mills (USN) with Captain T. A. Solberg (USN) as deputy, Dr. W. K. Lewis, and Dr. H. D. Smyth) received a multitude of suggestions from men on the various projects, principally along the lines of the use of nuclear energy for power and the use of radioactive by-products for scientific, medical, and industrial purposes. While there was general agreement that a great industry might eventually arise, comparable, perhaps, with the electronics industry, there was disagreement as to how rapidly such an industry would grow; the consensus was that the growth would be slow over a period of many years. At least there is no immediate prospect of running cars with nuclear power or lighting houses with radioactive lamps although there is a good probability that nuclear power for special purposes could be developed within ten years and that plentiful supplies of radioactive materials can have a profound effect on scientific research and perhaps on the treatment of certain diseases in a similar period.

"*Planning for the Future.* During the war the effort has been to achieve the maximum military results. It has been apparent for some time that some sort of government control and support in the field of nuclear energy must continue after the war. Many of the men associated with the project have recognized this fact and have come forward with various proposals, some of which were considered by the Tolman Committee, although it was only a temporary advisory committee reporting to General Groves. An interim committee at a high level is now engaged in formulating plans for a continuing organization. This committee is also discussing matters of general policy about which many of the more thoughtful men on the project have been deeply concerned since the work was begun and especially since success became more and more probable.

"*The Questions before the People.* We find ourselves with an explosive which is far from completely perfected. Yet the future possibilities of such explosives are appalling, and their effects on future wars and international affairs are of fundamental importance. Here is a new tool for mankind, a tool of unimaginable destructive power. Its development raises many questions that must be answered in the near future.

"Because of the restrictions of military security there has been no chance for the Congress or the people to debate such questions. They have been seriously considered by all concerned and vigorously debated among the scientists, and the conclusions reached have been passed along to the highest authorities. These questions are not technical questions; they are political and social questions, and the answers given to them may affect all mankind for generations. In thinking about them the men on the project have been thinking as citizens of the United States vitally interested in the welfare of the human race. It has been their duty and that of the responsible high government officials who were informed to look beyond the limits of the present war and its weapons to the ultimate implications of these discoveries. This was a heavy responsibility. In a free country like ours, such questions should be debated by the people and decisions must be made by the people through their representatives. This is one reason for the release of this report. It is a semi-technical report which it is hoped men of science in this country can use to help their fellow citizens in reaching wise decisions. The people of the country must be informed if they are to discharge their responsibilities wisely.".

"Progress in scientific research and development is an indispensable condition to the future welfare and security of the nation. The events of the past few years are both proof and prophecy of what science can do. . . .

"No nation can maintain a position of leadership in the world of today unless it develops to the full its scientific and technological resources. No Government adequately meets its responsibilities unless it generously and intelligently supports and encourages the work of science in university, industry, and in its own laboratories. . . .

"In order to derive the full profit in the future from what we have learned, I urge upon the Congress the early adoption of legislation for the establishment of a single Federal research agency which would discharge the following functions:

1. Promote and support fundamental research and development projects in all matters pertaining to the defense and security of the nation.

2. Promote and support research in the basic sciences and in the social sciences.

3. Promote and support research in medicine, public health, and allied fields.

4. Provide financial assistance in the form of scholarships and grants for young men and women of proved scientific ability.

5. Coordinate and control diverse scientific activities now conducted by the several departments and agencies of the Federal Government.

6. Make fully, freely, and publicly available to commerce, industry, agriculture, and academic institutions the fruits of research financed by Federal funds."

President HARRY S. TRUMAN *in his message to the Congress, September 6, 1945.*

Contributed Original Research

The Design of X-Ray Powder Cameras

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The x-ray diffraction tool in most common use is the powder camera, yet it is usually ill-designed. This paper presents constructive criticisms of current powder camera design, and describes briefly a series of instruments having desirable features. Among the issues discussed are: adjustments, recovery of adjustments, desirable diameters, important recording ranges, film arrangements, fitting the film to cylindrical form, design of direct beam systems, elimination of air-scatter, specimen attachment and adjustment, and scanning. The following special instruments are also discussed: high temperature cameras, large spacing cameras, and film cutters and punches.

INTRODUCTION

CAMERAS for making Debye-Sherrer type x-ray powder photographs are so simple in principle that they are frequently ill-designed with respect to efficiency and convenience. Poorly designed cameras may be wasteful of x-radiation and consequently require over-long exposures; they usually lack easy adjustments, lack provision for recovering adjustment, so that no two photographs are taken under duplicate conditions, usually do not make provision for reducing air-scatter and consequently produce fogged films, and, in general, are inconvenient to operate in a routine manner. Many of these criticisms can be justly made of most current commercial models as well as many home-made models.

In the present paper, constructive criticisms are advanced for various features of powder camera design, and models designed by the writer which avoid the faults discussed are presented. These cameras were displayed at the Ann Arbor Meeting of the American Society for X-Ray and Electron Diffraction and have been duplicated at several laboratories.

If the discussion of powder camera design given here appears to be overdone, it should be remembered that more actual time is spent with the powder camera than with any other x-ray diffraction tool, both in routine industrial and in academic investigations. In spite of this, commercial models have not been improved in any basic way for some time, and a guide should be available for such improvement.

CAMERA MOUNTING AND ADJUSTMENT

It is a common fault of powder camera mountings that they are difficult to adjust, and that after the powder camera is removed from its place at the x-ray tube, it does not automatically recover duplicate alignment with the x-ray beam when returned to position. These difficulties are overcome in the design shown in Figs. 1 and 2.

A unique feature of the camera mounting, as compared with commercial powder camera mountings, is that it is attached directly to the same support which holds the x-ray tube (Figs. 1 and 2). This arrangement prevents play between x-ray tube and camera, so that, once the pinhole system is aligned to the x-ray beam, the alignment is permanent.

The cylindrical camera is attached to a base with respect to which it has certain adjustments. These are provided by a shaft-like extension of

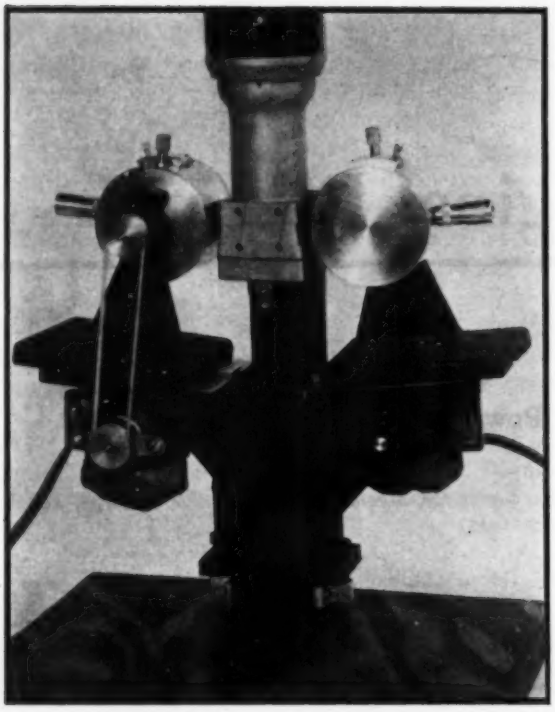


FIG. 1. Battery of 57.3 mm-diameter powder cameras, showing the general assembly of parts. The member supporting the camera cylinder is termed the *base*. This is keyed by means of a dovetail to a *bracket*, which, in turn, is keyed to the x-ray tube *support*.

the camera which fits into a horizontal bearing in the base. The bearing is split, and is normally kept tightly clamped to the shaft by means of a screw. When the screw is loosened, the camera shaft may be rotated or translated in the bearing. When the screw is tightened, the camera-base assembly is a rigid unit.

This assembly is attached to the x-ray tube base through a bracket, which offers certain adjustments. The camera rests on the approximately horizontal surface of the bracket, and is slipped onto it with the aid of a longitudinal dovetail which serves two functions. It fixes the orientation of the camera so that, once adjusted, it maintains alignment with the x-ray beam in the vertical plane, and it also prevents the camera from being accidentally knocked off the bracket. The dovetail can be loosened and its orientation on the horizontal surface of the bracket varied within small limits.

The vertical surface of the bracket rests against a corresponding surface on the x-ray tube base which is approximately normal to the x-ray beam. The two surfaces are held together by means of a single, central screw. A large vertical movement of the bracket on the base is permitted when this screw is loosened, and rotation about an axis normal to the surface is prevented by a vertical keyway between the surfaces.

The mounting, just described, permits the following motions of the pinhole axis with respect to the x-ray beam, during adjustment:

Translation in 3 orthogonal directions:

(1) *Left and right* (when the screw in the split bearing of the camera base is loosened).
(2) *Up and down* (when the screw in the x-ray tube base is loosened).
(3) *Parallel with the beam* (as the camera is pushed back and forth on the dovetail; the forward motion may be limited by an adjustable stop).

Rotations about 2 orthogonal axes perpendicular to the x-ray beam:

(1) *About a horizontal axis* (when the screw in the split bearing of the camera base is loosened).
(2) *About a vertical axis* (when the two screws holding the dovetail to the base are loosened).

It will be noted that to loosen the entire assembly for adjustment calls for loosening screws at only three points.

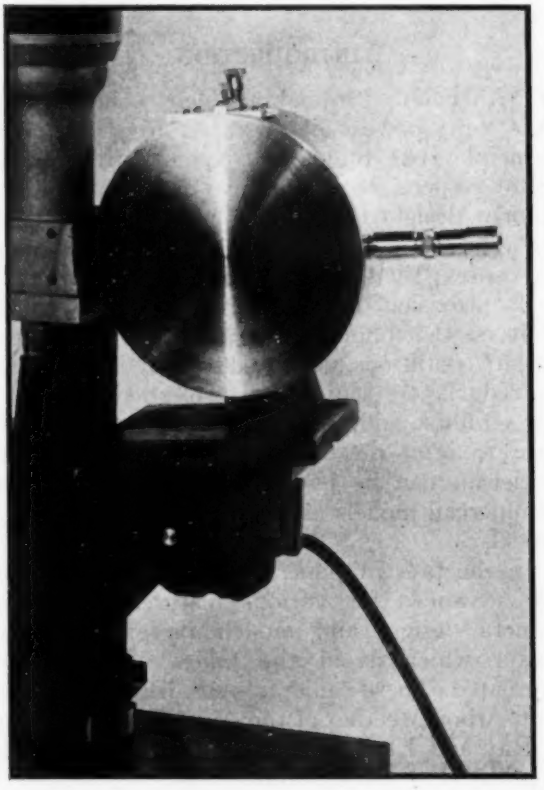


FIG. 2. 114.6 mm-diameter powder camera, in place against x-ray tube.

The initial adjustment of the camera is simply made and requires a few minutes time at most. A rough approximation is first made by removing the pinhole system of the camera and aligning the coarse beam which comes through the resulting hole. Final adjustments are made after the pinhole system is returned to the camera.

In practice, all screws may be just barely loosened so that all adjustments can be made by the use of a little force. The camera may then be manipulated as if free in space, and its position and orientation can be varied by hand until the spot on the fluorescent screen shows the pinhole system to be pointing at the focal spot of the x-ray tube. This gives approximate adjustment. For final adjustment, each motion is varied independently a little in order to find the position giving maximum intensity of the x-ray beam as judged by the brightness of the spot on the fluorescent screen. When these adjustments have been completed, the vertical translation coordinate of the bracket is fixed by a set screw. This screws in a lug in the bracket and rests on a shelf on the x-ray tube base.

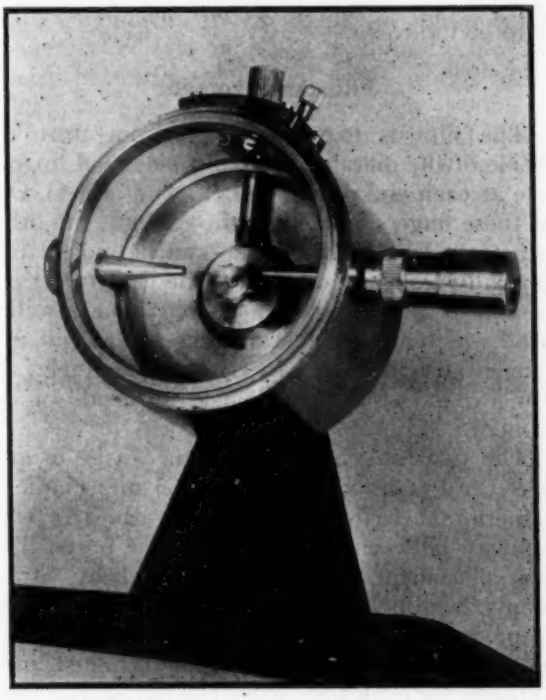


FIG. 3. View of interior of 57.3 mm-diameter powder camera. The pinhole system is contained in the cone at the left. Air scatter is reduced by enveloping the exit-beam with the cone at the right. The specimen holder is in the center, and can be pushed down, when eccentric, by the vertical pusher rod above it. This is operated by turning the large milled knob seen at the top of the cylinder. The film is expanded against the cylinder by circumferential pressure, applied to the film by the moveable finger, seen at the top right. The finger at the left is fixed.

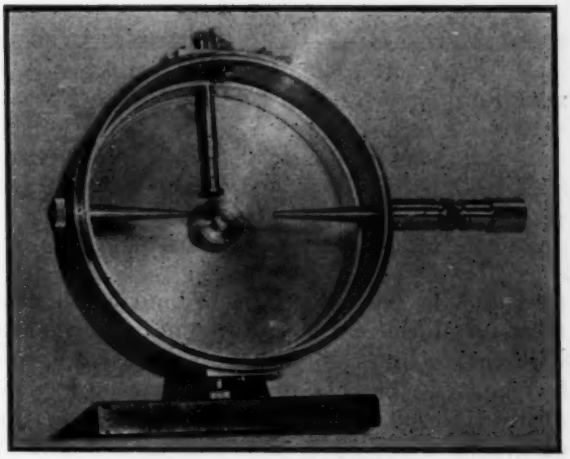


FIG. 4. View of the interior of the 114.6 mm-diameter powder camera.

Once adjusted, the camera-base-bracket-tube is a rigid unit whose only degree of freedom is the camera translation parallel to the x-ray beam; this permits removing and replacing the camera base with respect to the bracket. The entire attachment scheme has important advantages in the event that the x-ray port is required for other diffraction apparatus. Should this contingency arise, the entire adjusted assembly can be instantly removed as a unit by removing one screw, leaving the table top clear for other apparatus, and it can also be instantly replaced in correct adjustment whenever required. This is because the only degree of freedom between bracket and base, the vertical translation along the keyway, is not operative because of the set screw in the base. This was set after adjustment so that the vertical coordinate of the bracket is correct to align the pinhole system to the x-ray beam. When the bracket is returned, the alignment is recovered.

THE CAMERA

General Features

The photographic film is expanded against the internal wall of a hollow cylinder (Figs. 3 and 4). Since the film is exposed directly to the diffracted radiation, the camera must be inherently light-tight. This is partly effected by a screw-on cover, partly by light-tight entrance and exit ports for the x-ray beam. The cover screws on to the camera cylinder by comparatively coarse

TABLE I.

	1 millimeter on film corresponds with	
	arc, also deviation angle, 2θ	Bragg angle,
57.3 mm diameter camera	2°	1°
114.6 mm diameter camera	1°	½°

threads. The outer threads are omitted so that the axis of the cover is automatically aligned with that of the camera before the threads can be engaged. This feature prevents accidently misthreading the cover—an advantage in the darkroom. Both cover and camera are turned from bronze castings.

It is common practice to make cameras of arbitrary diameter. Arbitrary diameters are a convenience to a thoughtless manufacturer, but an unmitigated inconvenience to the user. There are good reasons why effective diameters of 57.3 mm and 114.6 mm should be used. For these diameters, the conversion factors from millimeters distance in the film to angle of arc is 2 and 1°/mm respectively. The film record can thus be measured with a standard millimeter scale, and no computation is required to obtain angles. The angular significance of the millimeter measurements on each of these camera diameters is expressed in Table I. When these camera diameters are used, the spacing, d, may be found immediately from each film reading by consulting tables[1] for converting θ to d for various values of wave-length.

Another advantage of these camera diameters is that commercial photographic films can be easily cut, without waste, to provide film strips for them. Thus, the circumference of the 57.3 mm-diameter camera is 180 mm. Seven inches is 177.8 mm. The commercial "seven" inches of the 5×7 inch x-ray film is a trifle less than seven inches (actual dimensions $4\frac{60\pm1}{64}\times6\frac{60\pm2}{64}$ inches[2]). Thus a 5×7 inch film can be cut into five approximately one-inch strips, each of which come just short of covering the circumference. For the 114.6 mm-diameter camera, either a 14-inch film strip may be used to cover the full circumference, or a 7-inch film may be used to cover any 180° range. The desirable 180° ranges are:

(*a*) One complete half of the double reflection record, i.e., the 180° range from exit port to entrance port, corresponding with a θ range from 0° to 90°.

(*b*) The forward reflection field only, i.e., 90° on each side of the exit port, corresponding with a θ range from 0° to 45°. This range is especially valuable for use with large cell organic crystals, such as soaps, which give few, if any, reflections beyond this range.

(*c*) The back reflection field only, i.e., 90° on each side of the entrance port, corresponding with a θ range of 45° to 90°. This range is necessary when carrying out precision spacing investigations.[3]

The 57.3 mm-diameter camera (Figs. 1 and 3) is desirable for routine identification, and, in general, for comparison work with inorganic crystals. The larger 114.6 mm-diameter camera (Figs. 2 and 4), of course, requires much longer exposures. It is a preferable size, however, when more complicated crystals are to be examined, or where the powder photograph, if complicated, is to be indexed.

Film Arrangement

The film is expanded directly against the inside of the metal cylinder by means of fingers, one at each end of the film (Figs. 3 and 4). One of these fingers is fixed to the camera, the other is attached to a slide which can be moved tangentially from the outside of the camera. It can be clamped in any position by turning a milled head which acts against a brake. When put into tangential compression by the movable finger, the film expands radially against the inside of the camera.

The two free ends of the film can be located in several positions. For the 57.3 mm-diameter camera, the writer favors having the free ends halfway between entrance and exit ports. This film placement was first suggested by Straumanis,[4] who pointed out that when this arrangement is used, the effective camera diameter, including shrinkage, can be determined individually for each film, since the diffraction rings about entrance and exit ports center at points differing by exactly half the circumference. The writer prefers the Straumanis film placement for this and two additional reasons: (1) The diffraction record is complete, without small missing regions either near 0° or 90° θ and (2)

[1] *International Tables for the Determination of Crystal Structures* (Gebrüder Borntraeger, Berlin, 1935) Vol. 2, pp. 585–608.

[2] "American Standard dimensions for industrial x-ray sheet film (inch sizes) Z 38.1.25—1944," J. Am. Opt. Soc. **34**, 626(1944).

[3] M. J. Buerger, *X-Ray Crystallography*, Chapter 20.

[4] See reference 3, pages 394–396.

a doubled back-reflection record is always available for precision determinations[5] either by the method of Cohen or by a modification of the method of Bradley and Jay.

The writer's 114.6-mm-diameter camera (Figs. 2 and 4) is arranged to accommodate, directly, a half-film (a 1×7 inch strip with central hole) in either the forward-reflection field, as seen in Fig. 4, or in the back-reflection field, or a full film (a 1×14-inch film with two holes, placed Straumanis-wise) covering the complete double field. The front and back half-fields are made possible by use of a film spreader and fixed grips near the top of the camera which face both ways, and a plug-in lower fixed grip. The latter is removable to permit use of full 14-inch film. The full record can also be covered in two 7-inch films without a missing region by using two 7-inch films, one in the front-reflection field and one in the back-reflection field, butted end to end. The complete single range from 0° to 180° can be realized with a 7-inch film by a similar arrangement. To do this, a spacer made of metal, or even a piece of dummy film, is placed between +90° and 180° in the back-reflection field and another between +90° and 0° in the front-reflection field. The useful film is spread between the ends of the two dummy films in the lower half of the camera.

Direct-Beam System

The collimating system of a powder camera is commonly placed in a tube which is external to the camera cylinder. This design increases the source-to-specimen distance by the length of the external collimator. Since the intensity of the x-ray beam falls off as the inverse square of the distance, it is obvious that such design implies an extravagant waste of x-radiation. There is no necessity for collimating the beam outside the camera; this can be adequately provided for within the interior of the camera, thus increasing the available x-ray intensity at the specimen. In the design suggested below, nothing but a small milled nut protrudes beyond the camera cylinder (Figs. 3 and 4).

In some powder camera designs, the direct beam is prevented from reaching the film by a lead cup placed in front of the film. This scheme has the disadvantage that the back scatter by the inside of the cup reaches the film in a large zone in the back-reflection region. A superior method[6] of getting rid of the beam is to permit it to escape from the camera through a hole. This requires a hole in the film if the film extends to the point $\theta=0$. This system has the disadvantage of bringing the source of air-scatter very near the film in the region of the hole and thus of producing films fogged by this scatter in the region of the hole. This is corrected in the design to be described. In this design, the x-ray beam is led to the specimen and conducted away from it within tubes which reach nearly to the specimen.

In the camera preferred by the writer, the direct x-ray beam is handled by means of two assemblies, the entrance-port assembly and the exit-port assembly. The first collimates the beam while the second performs several functions: it removes the direct x-ray beam, at the same time reducing air-scatter to the film, and it shows the centering of the beam and specimen on a fluorescent screen. These two assemblies fit into holes on opposite sides of the camera cylinder and are held in place by means of screw threads. When the film is used Straumanis-fashion so that the beam must enter and leave through holes in the film, both assemblies must be removed and returned in the darkroom when removing or loading a film.

The entrance-port assembly consists essentially of an internal nozzle-shaped tube, pointed toward the specimen. In the tube are mounted two lead disks having central holes which define the direct beam within the camera. The hole toward the x-ray source is the limiting pinhole. It is so located that it lies within the hole in the film and on an extension of the surface of the film cylinder. This location gives rise to well-focused back-reflection lines. The breadth of the lines in this region is equal to the diameter of the limiting pinhole. The second pinhole is near the specimen. It merely limits the aperture of the cone originating at the first pinhole, and its size is made only small enough to prevent this cone from spreading beyond the size of the exit hole in the exit-beam screen. A third, larger aperture removes the diffraction originating at the second pinhole.

After impinging on the specimen, the direct x-ray beam is ushered away from the central part of the camera by an exit-beam screen. This is essentially a tube and is an integral part of the exit-port assembly. Its function is to surround the beam as it traverses the distance between

[5] See reference 3, pages 426–432; 411–426, especially p. 420.

[6] M. J. Buerger, "An x-ray powder camera," Am. Mineralogist 21, 11–17 (1936).

specimen and the hole in the film and thus shield the film from radiation scattered by the air in the beam path. Cameras provided with such shields produce x-ray photographs which are comparatively clean in the region near $\theta=0$ (Figs. 6 *A* and *B*).

The parts of both entrance-port assembly and exit-port assembly which lie within the camera are shaped externally in the form of frustrums of cones (Figs. 3 and 4). The cones are coaxial and nearly point-to-point with their apices slightly to the near sides of the specimen and their bases outlined by the entrance and exit holes in the film. Shaped this way, these cones do not screen the film from any radiation diffracted by the specimen except that which would fall within the area of the two holes.

The exit-beam screen is an integral part of the exit-port assembly which conducts the direct beam through the exit hole in the film. This assembly is essentially a continuous tube which extends well outside the camera cylinder. It is removed, during the loading operation, by unscrewing it from the threaded hole in the camera cylinder.

The hole in the end of the tube is closed by a series of three disks of different material in this succession: (1) black paper, to make the camera interior light-tight; (2) fluorescent screen facing outward, so that the centering of the beam and specimen can be checked at any time; and (3) lead glass, to protect the operator when making such observations and to present the direct beam from leaving the camera and entering the room. These three disks are held against the end of the tube by a friction cap. The whole exit-port assembly acts as a unit and is inserted into the exit hole of the camera cylinder.

Specimen Attachment, Adjustment, and Rotation

Certain designs of powder camera either provide no adjustment for centering the specimen with respect to the axis of rotation; others provide more than is necessary. In the latter category is the powder camera equipped with complete goniometer head for adjusting angular and eccentricity errors of the specimen. Angular adjustment is unnecessary unless the specimen is to be scanned. The centering adjustment can be provided in a very simple manner.

In an earlier design,[6] the writer used a simple centering device with the aid of which are applied rectangular coordinate corrections to the position of the specimen by turning two adjusting screws. Unfortunately these screws were within the cylinder of the camera and gave rise to complaints from those with large hands. The improved design utilizes polar coordinate adjustment. To correct an error in concentricity, the rotation axis is turned until the error is directed up, then the specimen is pushed down. Details are given below.

The powder sample, prepared in the form of a tiny rod in one of a number of ways,[6,7] is normally attached to a short section of $\frac{3}{32}''$ metal rod, which serves as a handle in preparation and mounting. To mount this in the camera, it is merely grasped with small pliers or forceps and thrust into an axial hole in the centering device. Here it is retained in place by pressure from a weak spring made from very fine wire which protrudes into the hole from a longitudinal slot. The pressure is transmitted through a roller.

When the specimen has been thus mounted in the camera, the eccentricity of the specimen axis is observed through a very simple microscope[8] which is temporarily placed in the normal position of the entrance-port assembly. The microscope consists merely of a lens attached to the end of a small tube. When screwed in place on the camera, the lens is outside of the camera and the opposite end of the tube almost touches the specimen. The lens is focused on the specimen; consequently the eye sees the specimen framed in the far end of the tube. When the shaft is rotated, an eccentric specimen appears to execute an up-and-down motion as seen through the microscope. To correct the error, one rotates the shaft with the left hand until the specimen is at the top of its motion, then, with the right hand, he operates a pusher which displaces it down to the center of oscillation. The pusher is manipulated by turning a milled head which protrudes from the top of the camera. The screw motion is transmitted as a pure translation to a pusher rod keyed to the inside of a tube. (Figs. 3 and 4). The whole pusher assembly screws into a hole in the top of the camera cylinder. The pusher slides the plate holding the specimen over a plate attached to the shaft. The operation of centering with this device is the work of a moment. When the centering is perfect, the specimen appears unmoved when rotated.

[7] Joseph S. Lukesh, "An improved technique for mounting powdered samples for x-ray diffraction," Rev. Sci. Inst. 11, 200–201 (1940).

[8] M. J. Buerger, "The precision determination of the linear and angular lattice constants of single crystals," Zeits. f. Krist. (*A*) 97, 441 (1937).

The shaft which rotates the specimen works in two gusset-type disk bearings and is built accurately concentric with the interior cylindrical surface of the camera. This provision reduces eccentricity errors, inherent in different designs, to a negligible amount.

The specimen shaft is rotated through a pulley and rubber band drive by the smallest electric clock motor (Fig. 1). An alternative drive is friction coupling through one or two rubber disks. Either method effectively insulates the motor from the camera in the event of static discharges, still common in modern commercial x-ray tubes.

The motor is mounted on a rod which is clamped by means of a set screw to a sleeve in the camera bracket. The position of the pulley is thus easily adjusted by loosening one screw and changing the position of the shaft.

SPECIMEN-SCANNING ATTACHMENT

By rotating the specimen, less spotty diffraction lines are produced than if the specimen were stationary; rotation is necessary for any except the finest-grained samples. When the sample is very coarse grained, even rotation does not suffice to bring enough different grains into reflecting position to produce continuous diffraction lines. Barrett and Guy[9] have pointed out that by utilizing a translation as well as a rotation motion, the number of grains brought into condition to reflect is greatly increased and better photographs are produced. Barrett suggested using a screw instead of a simple rotating shaft, thus giving the specimen a screw motion. This arrangement has the disadvantage of requiring coordination between the pitch of the screw and the timing of the photograph because the screw must not advance more than the length of the specimen during the exposure time. Since exposure times vary with symmetry, absorption, scattering power, etc., this provides a very inflexible arrangement.

An attachment which accomplishes the scanning without the drawbacks of the simple thread is shown in Fig. 5. It works on the following principle: The motor drives a shaft which carries two gears of almost identical size but one of which has one more tooth than the other. One gear meshes with a gear on a cam attachment, the other with a gear on the shaft. The latter rotates the shaft in the normal way, and the other rotates a position finger in a cam. If both gear ratios were identical, no translation would result. The differential ratio, however, turns the cam in the follower and slowly translates the shaft back and forth. About thirty rotations are made before one translation cycle is completed.

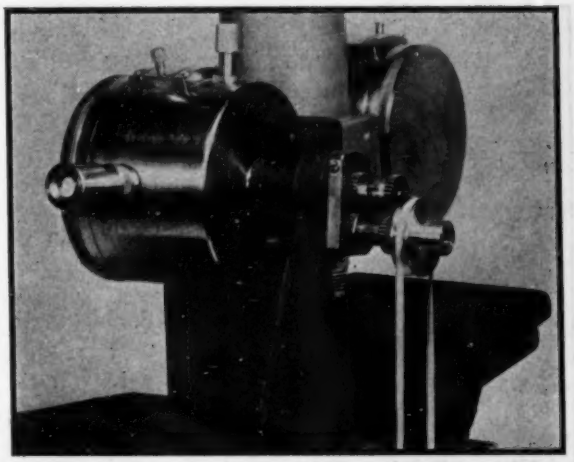


FIG. 5. Specimen-scanning device attached to a 57.3 mm-powder camera. The two lower gears turn with the drive pulley. The left pair of gears rotates the specimen in the usual way; the right pair works the finger in the cam groove. Slightly different gear ratios give the upper gears a slow differential motion, The finger is attached to the specimen shaft, and the motion of the finger causes the specimen to travel left and right. This scanning can be dispensed with at will by merely removing the pulley and replacing it, left for right. This change substitutes a blank shaft for the lower gear, so that the finger has no motion with respect to the cam groove. The entire attachment may be removed by loosening the screw seen at the upper left of the attachment, after which the normal drive pulley may be substituted, as in Fig. 1.

To use any scanning device, it is vital that the specimen be straight and accurately parallel with the rotation axis in order that, when translated, it may remain centered in the beam. Without scanning it is sufficient that a short section of the specimen be centered.

Examples of powder photographs made with the same sample stationary, rotated, and both rotated and translated, are shown in Fig. 6*C*, *D*, and *E*.

FILM CUTTER AND PUNCH

When much routine powder-diffraction work is undertaken, films must be cut and perforated easily and accurately. Both of these operations are accomplished with the device shown in Fig. 7. This is a modification of the ordinary photographic paper trimmer, fitted with an outside stop so that it cuts one-inch strips. Projections on the cutter blade push plungers which perforate the film in the manner of a ticket punch. The cutting ends of the plungers are sharpened

[9] C. S. Barrett and A. G. Guy, "Debye-type x-ray camera for coarse-grained material," Rev. Sci. Inst. 15, 13 (1944).

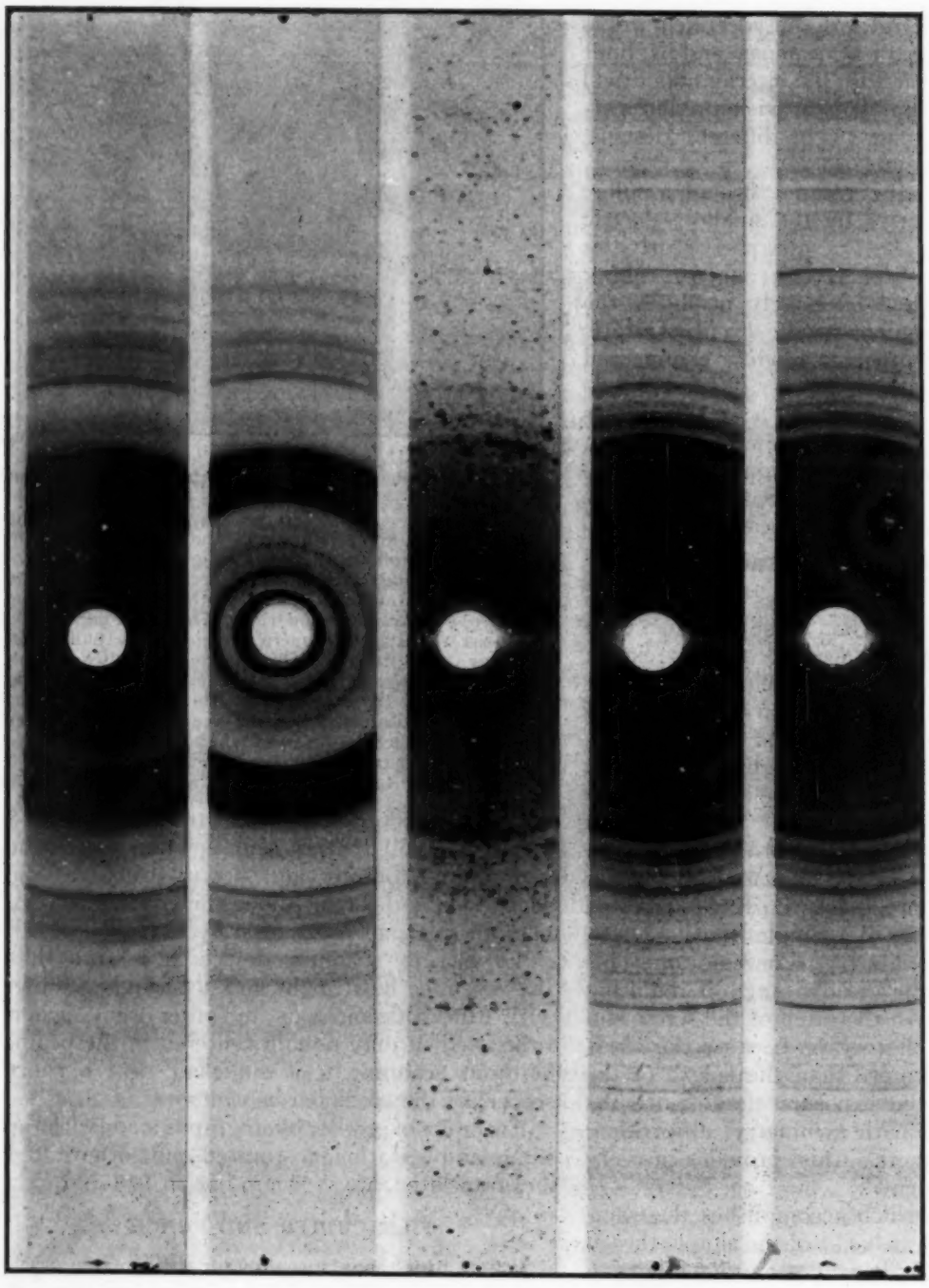


FIG. 6. X-ray photographs illustrating text discussions. *A* and *B* are photographs made without and with exit-beam screen, showing how its use eliminates air-scatter (the acid soap, β HSt·NaSt; Cu$K\alpha$, 1½ hours).

C, *D*, and *E* are photographs made with various motions of the specimen, illustrating the advantages of scanning: *C* is with neither rotation nor translation; *D* is with rotation only; and *E* is with both rotation and translation. The central dark area in these three photographs is due to general radiation scattered by the specimen, as evidenced by the Laue spots in the central region of *C*. (Adularia feldspar, $KAlSi_3O_8$, monoclinic, ground to 100 mesh and finer; Cu$K\alpha$, 2 hours.)

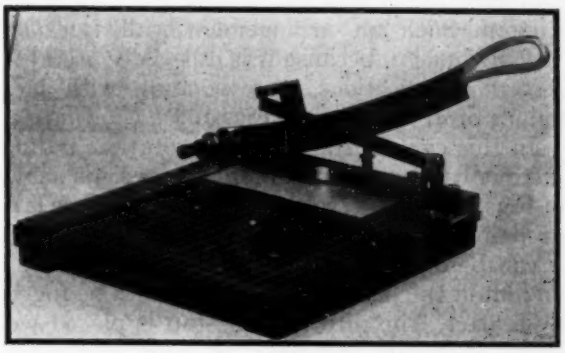
in saddle shapes. When a one-inch strip is being cut, the part of the remainder which will be the next strip to be cut is simultaneously perforated. A universal perforator has three positions. When a plunger is placed in the central position, a film strip is produced with a hole in the center. When a plunger is placed in each of the outer positions, a film strip is produced for the Straumanis arrangement.

POWDER CAMERAS FOR SPECIAL PURPOSES

High Temperature Camera

The camera herein described may be adapted

to high temperature diffraction investigations.[10,11] A photograph of the 57.3 mm-diameter camera is shown in Fig. 8. The power source for heating such cameras has been much improved by Dr. Frank G. Chesley and consists of a voltage stabilizer, stepdown transformer, and wattmeter in sequence. This source has been used by both Dr. Chesley and the writer on separate apparatus with much success.



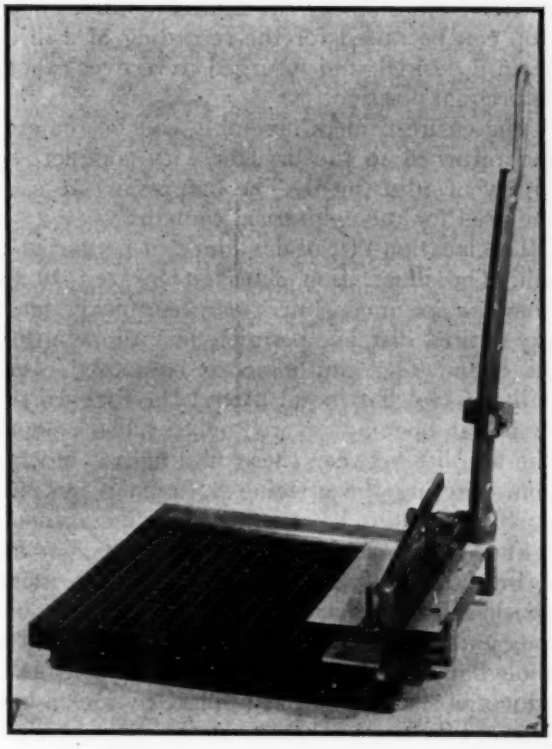


FIG. 7. Device for cutting films of correct width, and punching them with holes located exactly for insertion of entrance-port and exit-port assemblies. The punches are seen here set up for providing holes for the Straumanis arrangement of film.

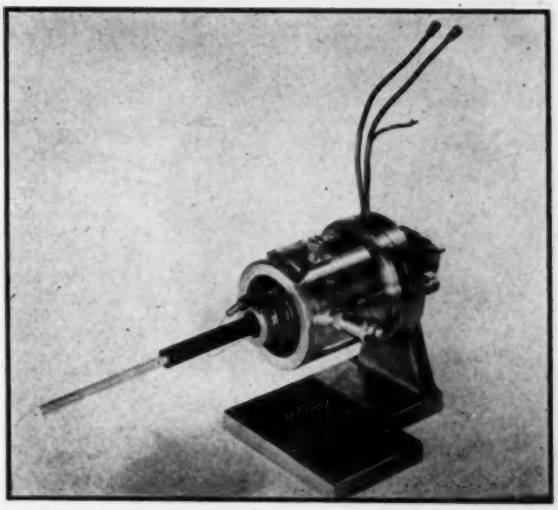


FIG. 8. High temperature x-ray powder camera.

The high temperature camera is useful not only in making studies of polymorphic substances and their transitions, but also in carrying out complete phase diagram investigations.[11] These investigations are preferably conducted with the aid of temperature-controlled cameras rather than by the customary quenching technique, since the latter method provides merely the patterns of phases descending from those being investigated, not necessarily patterns of the original phases themselves.

Half-Record Camera

Figure 9 shows a photograph of a camera intended to record only half the total possible record. The model illustrated is specifically designed to record only the forward-reflection 180° range of 2θ, centering around $2\theta=0$. Such a range is sufficient for most identification purposes and for routine investigations of complicated organic crystals whose back-reflection records are weak or substantially absent. The advantage of the partial cylinder construction is that the target-to-sample distance is reduced and consequently exposures are reduced.

In this design, the cover consists of a plate which is recessed so as to produce a light-maze as it fits against the partial cylinder of the camera. The inside of the plate carries two projections which fit into the main body of the camera and prevent the plate from falling off.

[10] M. J. Buerger, Newton W. Buerger, and Frank G. Chesley, "Apparatus for making x-ray powder photographs at controlled, elevated temperatures," Am. Mineralogist **28**, 285–302 (1943).

[11] Newton W. Buerger, "The chalcocite problem," Ec. Geol. **36**, 19–44 (1941).

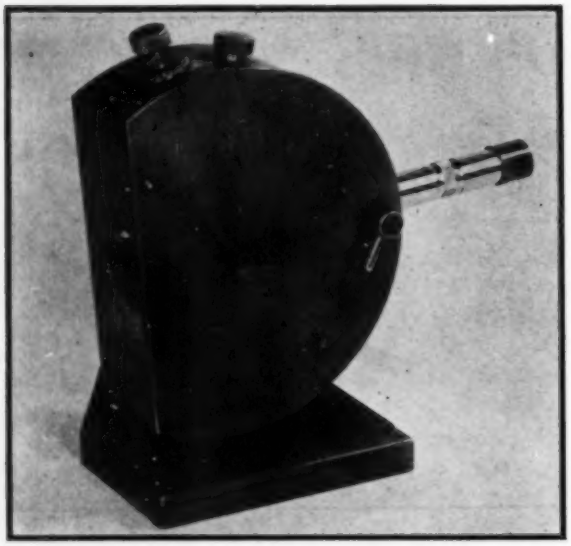


FIG. 9. Half-record x-ray powder camera.

One of these projections is at the center of the straight side, and is permanent. The other is a cam lock, operated by the key seen at the right of the cover.

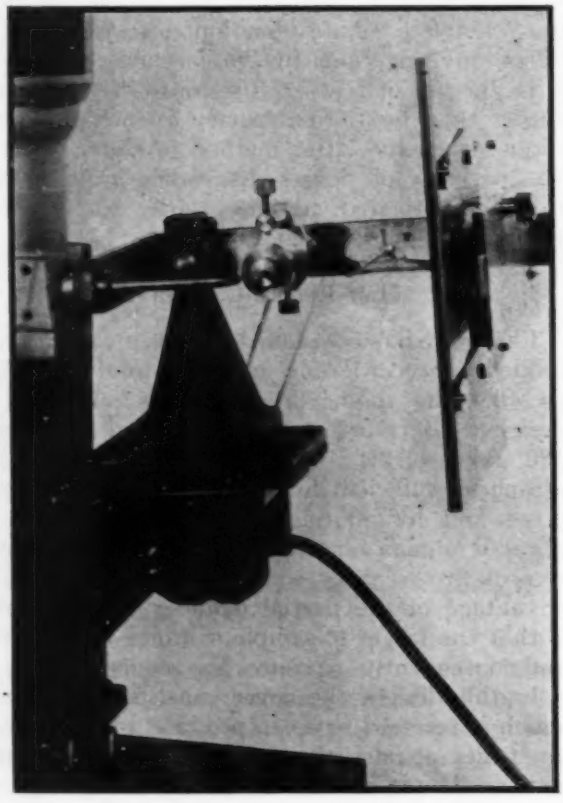


FIG. 10. X-ray powder camera for recording large spacings.

Large Spacing Camera

When it is necessary to work with greater dispersion than that of the 114.6 mm-diameter camera, namely 1 mm/degree for 2θ, it can be attained without sacrificing any of the advantages of the general design already discussed, by replacing the cylindrical camera with a flat-film camera. Such an arrangement, called a *large spacing camera*, because it is especially useful for recording reflections corresponding with large values of d, is shown in Fig. 10.

In this camera the specimen rotation and adjustment follow substantially the designs shown in Figs. 3 and 4. In this case, however, there is no need to lead the direct beam through the film; it can be blocked by a beam stop since there is no film in the back-reflection region to be fogged by scatter. The direct-beam stop is placed well ahead of the film, however, in order to eliminate scatter from air near the film. The direct-beam stop can be raised for the recording of a direct beam if desired, and returned to recover exactly its original position.

The entire camera assembly can be removed and returned to the bracket with complete recovery of adjustment. The bracket is the same one used for the cylindrical cameras.

The location of the flat film can be varied at will. The illustration shows it located 10 cm from the specimen, but it can be quickly set at any desired distance permitted by the length of the beam. The film-holder is composed essentially of two flat metal plates, the forward one having an opening covered only by black paper. The film fits between these, and light is blocked from entering the opening between plates and reaching the film, by means of four metal fins at right angles to the plane of the opening. The fins fit into slots in the plates and key the plates together. The fins are permanently fixed to one plate and fit loosely in the slots of the other. The whole flat cassette fits to the beam by means of a dovetail slide, and locks in place by a cam lock operated by the knob seen at the extreme right of the photograph.

ACKNOWLEDGMENT

The apparatus illustrated in this paper was constructed by Mr. Charles Supper. The writer is also indebted to Mr. Supper for many suggestions of a mechanical nature which have contributed toward making the several designs fulfil their desired functions.

Extension of the Beam Theory to Tapered Beams and Conical Shells

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(Received April 27, 1945)

The object of this paper is to extend to tapered beams and conical shells (membranes) the conventional formulas of the beam theory. This extension is made for two conditions: the plane tapered beam and the thin conical shell. The method of solution and the results obtained in the two cases are essentially different. The general analysis of the conical shell leads to solutions which are more complex than the corresponding solutions for cylindrical shells. However, in the case of applied stress analysis, simplifications are possible, which lead to solutions of the same type as for cylindrical shells.

INTRODUCTION

IT is generally assumed that the conventional formulas for normal stress distribution in cylindrical beams can be applied, without appreciable errors, to any shapes of non-cylindrical beams. When this simplification is introduced, normal stresses are readily determined, and shearing stresses are deduced by considerations of equilibrium. In two previous papers,[1,2] the author has examined problems of this type, in the two cases of developable and non-developable shells.

The object of the present analysis is to determine exact normal and shear stress distributions in the two special cases of tapered plane beams of constant thickness and of conical shells.

These distributions are exact in the same sense as beam distributions. They presuppose that the applied loads are distributed in such a manner that they are in equilibrium with the theoretical stresses. By application of Saint Venant's principle, it can be added that local modes of distribution of applied loads do not appreciably affect stress distributions at great distances from the applied loads. It must be noted, however, that Saint Venant's principle has a much smaller field of application for conical beams than for cylindrical beams: When the angle of the tapered beam or the solid angle of the cone is small, the structure can generally be assimilated to a cylindrical beam, at least in order to determine normal stresses, and Saint Venant's principle is applied to the equivalent cylinder. When the angle is great, transversal and longitudinal dimensions may be of the same order of magnitude, and Saint Venant's principle cannot be applied. Thus, Saint Venant's principle can be practically applied only when loads are concentrated near the apex of a structure and their effect on sections at a great distance from the apex is investigated.

In addition, it is necessary to modify Saint Venant's principle in order to apply it to membranes. This modification results from the following considerations. It is known that a system of loads in equilibrium which act on a small region of a solid body does not produce appreciable stresses in the portions of the body which are at a considerable distance from the region where the loads are applied. This result does not hold in the case of membranes, because, under certain loading conditions, the equilibrium between applied forces may require the transmission of stresses on the whole membrane.

A typical case has been examined by the author in connection with the analysis of cylindrical shells.[3] It has been shown that stresses in a cylindrical open shell are not completely determined in terms of the resultant of the applied forces and the resulting moment of these forces, taken in respect to an arbitrary reference point. To the two vectors, resultant and resulting moment in respect to a point, a new concept must be added, which has been called a bicouple. The bicouple acting on a section is a definite function of the normal stress distribution at that section. Generally, bicouples are due to the action of a torque (or of the moment of shearing forces in respect to the shear center); in this case, they are equal to the product of the torque by the length of the shell along which the torque is acting, this product being called the bimoment of the torque. Normal stresses in a cylindrical membrane can be calculated in terms of an applied bicouple in the same conditions as they

[1] L. Beskin, "Theory of Membranes in Shape of Developable Surfaces," J. App. Phys. 15, 547 (1944).
[2] L. Beskin, "Stress Analysis of Non-Developable Semi-Monocoque Structures," J. Aeronautical Sci. 12 (April 1945).
[3] L. Beskin, "Stress Analysis of Open Cylindrical Membranes," J. Aeronautical Sci. 11 (October 1944).

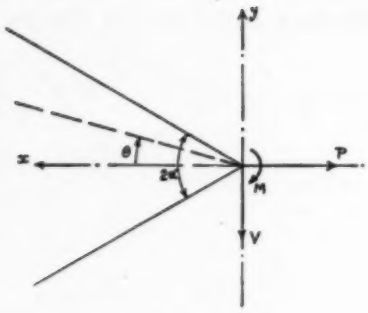

FIG. 1.

are conventionally calculated in terms of an applied axial force and couple; thus, when two systems of applied forces produce the same stresses, not only must they be statically equivalent, but in addition, they must correspond to the same bicouple. This condition of equal bicouple must be considered as a supplementary condition of equivalence for cylindrical membranes. A similar condition appears, as it will be shown hereafter, in the case of conical membranes.

The case of cones in form of a surface of revolution may be examined independently, because it requires a much simpler method of analysis than the general case. This case can be used as a basis for approximate methods of analysis of conical surfaces of more general shape.

I. PLANE TAPERED BEAM

The solutions corresponding to an axial load P and a tangential load V applied at the apex of a tapered beam are known.[4] Call (Fig. 1) 2α = angle of the beam, θ, r = polar coordinates, and N, T = normal stress flow acting in a radial direction and shear flow acting along the radius and the tangential direction (stress flow = stress × plate thickness). The solutions are:

Axial load P

$$N=[P\cos\theta]\Big/\left[r\left(\alpha+\frac{\sin 2\alpha}{2}\right)\right];\quad T=0. \quad (1)$$

Tangential load V

$$N=[V\sin\theta]\Big/\left[r\left(\alpha-\frac{\sin 2\alpha}{2}\right)\right];\quad T=0. \quad (2)$$

In these two cases, the normal stress distribution is somewhat different from the distribution in a beam with parallel flanges. For instance, under tangential load, if rectangular coordinates are used, the maximum normal stress flow calculated for the beam with parallel flanges is

$$N_\alpha'=\frac{6Vx}{(2y)^2}=\frac{3V\cos\alpha}{2r\sin^2\alpha},$$

instead of

$$N_\alpha=[V\sin\alpha]\Big/\left[r\left(\alpha-\frac{\sin 2\alpha}{2}\right)\right].$$

The ratio N_α/N_α' is equal to:

$$\frac{N_\alpha}{N_\alpha'}=\frac{4\sin^3\alpha}{3\cos\alpha(2\alpha-\sin 2\alpha)}=\frac{2\tan\alpha(1-\cos 2\alpha)}{3(2\alpha-\sin 2\alpha)}.$$

For small angles α it has the approximate value

$$N_\alpha/N_\alpha'=1+\alpha^2/5.$$

The relative differences are indicated on line 2 of Table I.

TABLE I.

	2α	30°	45°	60°	90°
Load V	$\frac{N_\alpha-N_\alpha'}{N_\alpha'}$	1.41%	3.31%	6.22%	16.80%
Couple M	$\frac{N_\alpha-N_\alpha'}{N_\alpha'}$	−4.06%	−9.01%	−15.70%	−33.33%

In order to define stress distribution for any loading condition, it is necessary to superpose a solution corresponding to a pure couple to the two solutions (1) and (2).

When a couple M is applied to the apex, the stress distribution can be defined in terms of a stress function ϕ,[5]

$$\phi=a\theta+b\sin 2\theta. \quad (3)$$

Expressing that the shear flow is equal to zero for $\theta=\pm\alpha$, and that the moment of the shear flows acting along a circumference is equal to M in respect to the apex

$$2\int Tr^2d\theta=-M,$$

it is found

$$\left.\begin{aligned} N&=\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}=\frac{2M\sin 2\theta}{r^2(\sin 2\alpha-2\alpha\cos 2\alpha)},\\ T&=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)=\frac{M(\cos 2\alpha-\cos 2\theta)}{r^2(\sin 2\alpha-2\alpha\cos 2\alpha)}. \end{aligned}\right\} (4)$$

[4] S. Timoshenko, *Theory of Elasticity* (New York, 1934), p. 93.

[5] See reference 4, page 114.

The maximum normal stress flow is

$$N_\alpha = 2M/r^2(1-2\alpha\cot 2\alpha),$$

instead of the value defined by the conventional beam theory

$$N_\alpha' = 3M/2r^2\sin^2\alpha.$$

The ratio N_α/N_α' is equal to

$$\frac{N_\alpha}{N_\alpha'} = \frac{4\sin^2\alpha}{3(1-2\alpha\cot 2\alpha)} = \frac{2(1-\cos 2\alpha)}{3(1-2\alpha\cot 2\alpha)},$$

or approximately

$$N_\alpha/N_\alpha' = 1-3\alpha^2/5.$$

The relative differences are given by the third line of Table I. When a load V_0 is applied at a section $r=r_0$, the stresses are obtained by superposition of two unit solutions $V=V_0$ and $M=-V_0r_0$. The normal stress distributions at the section $r=r_0$ are different from zero, since the two unit solutions do not correspond to opposite distributions at that section. This result is in contradistinction with the corresponding result for beams, for which there are no normal stresses at an end section to which a shearing force is applied.

The value of these normal stresses acting along the edge is obtained by difference of lines 2 and 3 of Table I. In this form, the stresses are given as a percentage of the stresses which would exist in a beam with parallel edges under the action of a couple V_0r_0. For instance, if $2\alpha=30°$, i.e., if the distance r is approximately equal to twice the depth of the beam, the local stresses reach 5.5 percent of the beam stresses defined above.

In general, applied forces do not satisfy these stress distributions, and Saint Venant's principle must be applied in order to justify the existence of these distributions at some distance from the applied loads.

II. CONICAL SHELLS WITH SPHERICAL SECTIONS

1. Conditions of Membrane Equilibrium of Conical Shells

The shells considered hereafter are thin, so that all stresses may be considered as acting in the planes tangent to the middle surface of the shell membrane equilibrium. Under given applied edge loads, these shells are generally statically determined. When these shells are stiffened by intermediate frames, the stress distribution must be defined by condition of compatibility of deformation.

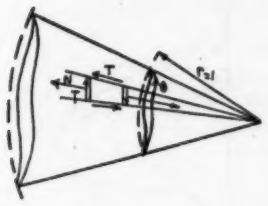

FIG. 2.

The sections which correspond to plane sections of a cylinder are defined for a cone by the intersections with spheres, the centers of which coincide with the apex of the cone (Fig. 2). In this manner, spherical directrixes are obtained, which are perpendicular to the generators, and normal stresses parallel and perpendicular to the generators, together with the corresponding shears, can be defined. Assuming that there are no surface loads, there are no normal stresses in the direction of the directrixes, and the state of stress is defined by N, normal stress flow along generators, and T, shear flow along generators and directrixes. Stress flows, rather than stresses are used in this analysis, in order to avoid the presence of the thickness in the equations of equilibrium.

The same equation of equilibrium as in the plane can be written between N and T, so that a stress function ϕ can be used and, with the same notations as in the plane (θ=angle between two generators, measured on the developed surface, r=distance to the apex), Eqs. (4) defining N and T as functions of ϕ can be applied. Since there are no normal stresses perpendicular to the generators, $\partial^2\phi/\partial r^2=0$, and ϕ is a function of first degree in r

$$\phi=\phi_0+\phi_1 r, \tag{5}$$

so that

$$N=\frac{\phi_0''}{r^2}+\frac{\phi_1+\phi_1''}{r}; \quad T=\frac{\phi_0'}{r}, \tag{6}$$

in which primes indicate differentiation with respect to θ. Thus, the solutions split into two distinct groups:

Group I. $N=\psi/r^2$; $T=0$. (7)

Group II. $N=\phi/r^2$; $T=\phi/r^2$. (8)

2. Statically Determinate Problems in Conical Shells

The determination of the state of stress in a conical shell under given applied edge loads is

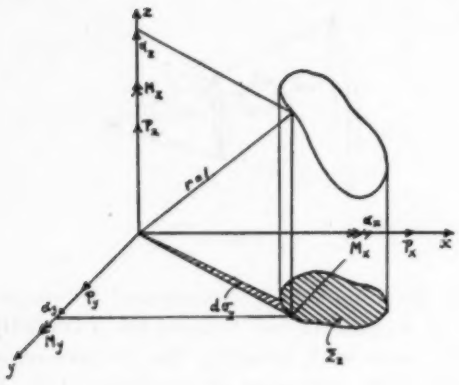

FIG. 3.

equivalent to the determination of the two functions ψ and ϕ.

Various conditions of edge loading may occur.

Condition 1.—The normal stresses and the shears are defined along a directrix. Eqs. (7) and (8) define ϕ and ψ:

$$\phi = T_a r_a^2; \quad \psi = N_a r_a - \phi'/r_a. \tag{9}$$

In these equations, index (a) represents the directrix along which the normal flow N_a and shear flow T_a are known.

Condition 2.—The shell is limited to two directrixes (radii r_a and r_b), which can be considered as fixed rigid bodies and transmit arbitrary shears. The normal stress flows N_a and N_b along the two directrixes are known. Using expressions (7) and (8) to represent stresses, the conditions of equilibrium at the edges are

$$N_a = \psi/r_a + \phi'/r_a^2; \quad N_b = \psi/r_b + \phi'/r_b^2. \tag{10}$$

Solving these equations, ψ, ϕ and the stress flows are found.

$$\psi = \frac{N_a r_a^2 - N_b r_b^2}{r_a - r_b}; \quad \phi' = \frac{N_a r_a - N_b r_b}{(1/r_a) - (1/r_b)}. \tag{11}$$

$$\left.\begin{aligned} N &= \frac{N_a r_a^2 - N_b r_b^2}{r(r_a - r_b)} + \frac{N_a r_a - N_b r_b}{r^2((1/r_a) - (1/r_b))}; \\ T &= \frac{1}{r^2((1/r_a) - (1/r_b))} \int (N_a r_a - N_b r_b) d\theta. \end{aligned}\right\} \tag{12}$$

The integral in the second Eq. (12) must satisfy definite boundary conditions. If the shell is open, i.e., is limited by two generators, the shear flow must be equal to zero along each edge generator which means that (the index d indicates that the integrals are taken along the complete directrix)

$$r_b \int_d N_b d\theta = r_a \int_d N_a d\theta = R. \tag{13}$$

If the shell is closed, the same shear flow must be found when a complete integration is effected along the directrix, which means that condition (13) also applies to this case.

The quantity R defined by Eq. (13) will be called "conical resultant." Equation (13) means that the conical resultant is independent of the radius. When a cylinder is considered instead of a cone, Eq. (13) simply means that the axial load is the same at the two ends of the cylinder. In the case of a cone of revolution (angle α between the generators and the axis), the conical resultant is equal to the axial load divided by $\cos \alpha$. In these two cases, condition (13) is a conventional condition of equilibrium.

Condition 3.—The shell is limited to two directrixes, which can redistribute shears; the loads to be transmitted by the shell and the normal stress flows along the two directrixes are known.

The relations found for Condition 2 apply in this case. In addition, it is necessary to verify that the shell is in equilibrium under the applied loads, defined by the components P_x, P_y, P_z along 3 axes and the moment components M_x, M_y, M_z in reference to the apex of the shell (Fig. 3).

Call $\alpha_x(\theta)$, $\alpha_y(\theta)$, $\alpha_z(\theta)$ = the direction cosines of the generators; $d\sigma_x(\theta)$, $d\sigma_y(\theta)$, $d\sigma_z(\theta)$ = the projections of the elemental areas of the lateral surface of the cone, limited by two generators and the directrix at the unit distance from the apex

$$\left.\begin{aligned} d\sigma_x = \alpha_y d\alpha_z - \alpha_z d\alpha_y; \quad d\sigma_y = \alpha_z d\alpha_x - \alpha_x d\alpha_z; \\ d\sigma_z = \alpha_x d\alpha_y - \alpha_y d\alpha_x; \end{aligned}\right. \tag{14}$$

σ_x, σ_y, σ_z = the projections of the lateral surface of the cone, limited by an edge generator (open shell) or a fixed generator (closed shell), a varying generator and the unit directrix.

The conditions of equilibrium are three equations of equilibrium of forces, of the type

$$\int_d N(r d\theta)\alpha_x + \int_d T(r d\alpha_x) = P_x, \tag{15}$$

and three equations of equilibrium of moments, of the type

$$2\int_d T r^2 d\sigma_x = M_x. \tag{16}$$

Substitution of N and T in terms of ϕ and ψ (Eqs. (7) and (8)) leads to conditions of the type

$$\int_d \psi\alpha_z d\theta + \frac{1}{r}\int_d (\phi'\alpha_z d\theta + \phi d\alpha_z) = P_z, \quad (17)$$

$$2\int_d \phi d\sigma_z = M_z. \quad (18)$$

The second integral in Eq. (17) is

$$\int_d d(\phi\alpha_z) = \phi\alpha_z\Big]_0^d$$

which is equal to zero: for an open shell, ϕ is equal to zero at both ends and for a closed shell, ϕ is a univalued function (condition 13).

Substitution of expressions (11) into Eqs. (17) and (18) leads to conditions of the type

$$\int_d \frac{N_a r_a^2 - N_b r_b^2}{r_a - r_b} = P_x; \quad (19)$$

$$\frac{r_a r_b}{r_b - r_a}\int_d 2d\sigma_z \int (N_a r_a - N_b r_b)d\theta = M_z. \quad (20)$$

Equation (20) can be transformed by integration by parts. For an open shell, the shear at the edge generators is equal to zero; as this shear is represented by the indefinite integral in Eq. (19), there is no constant term in the result of the integration by parts, and the following equality can be written, together with two similar relations

$$\int_d 2\sigma_z(N_a r_a - N_b r_b)d\theta = M_z(1/r_b - 1/r_a). \quad (21)$$

For a closed shell, the same integration by parts leads to

$$2\sum_z c - \frac{r_a r_b}{r_b - r_a}\int_d 2\sigma_z(N_a r_a - N_b r_b)d\theta = M_z \quad (22)$$

and two similar equations. In these equations the origins of areas σ_x, σ_y, σ_z are arbitrary because, due to relation (13), the first members are independent of these origins; Σ_x, Σ_y, Σ_z are the areas enveloped by the projections of the unit directrix on the three planes of coordinates; the arbitrary constant C is the constant of integration in the second Eq. (11), and defines a constant shear flow C/r^2 which corresponds to the three components of a couple M:

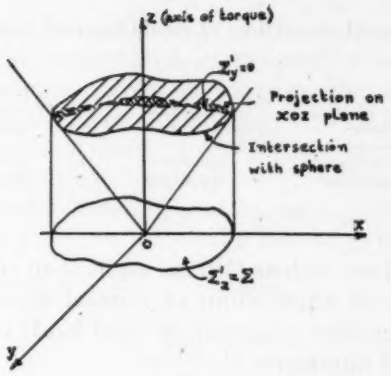

FIG. 4.

$$M_x = \int 2Tr^2 d\sigma_z = 2C\Sigma_x; \quad M_y = 2C\Sigma_y; \quad M_z = 2C\Sigma_z. \quad (23)$$

This couple is due to shear only and will be called a torque. Introducing the area Σ and the direction cosines ω_x, ω_y, ω_z,

$$\Sigma = (\Sigma_x^2 + \Sigma_y^2 + \Sigma_z^2)^{\frac{1}{2}}; \quad \omega_x = \frac{\Sigma_x}{\Sigma}; \quad \omega_y = \frac{\Sigma_y}{\Sigma}; \quad \omega_z = \frac{\Sigma_z}{\Sigma}, \quad (24)$$

the value of the torque is

$$M = (M_x^2 + M_y^2 + M_z^2)^{\frac{1}{2}} = 2C\Sigma, \quad (25)$$

and its direction is defined by direction cosines such as M_x/M which are precisely the quantities ω_x, ω_y, ω_z. A new system of axes of coordinates may be defined, the axis z' being directed along the vector representing the torque, which is the direction $(\omega_x, \omega_y, \omega_z)$ and the two other axes (x', y') being perpendicular to z'. The direction z' will be called the axis of torque of the shell; in the new system of coordinates (Fig. 4),

$$\Sigma_x' = \Sigma_y' = 0; \quad \Sigma_z' = \Sigma_z. \quad (26)$$

Thus, Eq. (22) and the two similar equations become

$$2\Sigma C - \frac{r_a r_b}{r_a - r_b}\int_d 2\sigma_z'(N_a r_a - N_b r_b)d\theta = M_z'; \quad (27)$$

$$\left.\begin{aligned} \int_d 2\sigma_x'(N_a r_a - N_b r_b)d\theta &= M_x'\left(\frac{1}{r_b} - \frac{1}{r_a}\right), \\ \int_d 2\sigma_y'(N_a r_a - N_b r_b)d\theta &= M_y'\left(\frac{1}{r_b} - \frac{1}{r_a}\right). \end{aligned}\right\} \quad (28)$$

TABLE II. Conditions of equilibrium for membranes.

Type of membrane	Cone with spherical frames	Cylinder, cone of revolution, cone with plane frames
Open membrane	4 conditions	1 condition
Closed membrane	3 conditions	0 condition

Equations such as (19) are equivalent to the conditions of equilibrium of normal stresses on a cylinder (one equation of axial load, two equations of moments).

Examining Eqs. (27) and (28) for a closed cone, and equations such as (21) for an open cone, it appears that there are 3 special conditions of equilibrium in the first case, and 2 conditions in the second case, since Eq. (27) replaces the conventional equation for torque. In addition, condition (17) must be satisfied in both cases.

When an open cylinder is considered,[3] the three equations such as (21) are reduced to one equation, due to the fact that the directrix of a cylinder is a plane curve. When the cylinder is closed, there is no condition of the type of Eq. (21). The comparison between the two types of membranes can be summarized in Table II. The cones of revolution and, more generally, the cones with plane frames have been classified together with the cylinders. This results from the analysis made in Section III, which shows that with a proper choice of stress components for cones with plane frames and with the conventional stresses for cones of revolution, the differential elements in equations such as (19) and (21) are proportional, and the two equations are not independent. The direction of the axis z being perpendicular to the plane of the frame, this result is valid for the two equations containing M_x and M_y. In addition, it can be shown that the condition of conical resultant, Eq. (13), can be reduced to the equation of equilibrium of forces along axis z. Thus, three conditions must be dropped from the second column of Table II, in order to obtain the third column, the results of which are common for cylinders, cones of revolution, and cones with plane directrixes.

In the case of a general cone with spherical frames, the three equations such as (21), or the two Eqs. (28) can be expressed in a concise manner by introducing the concept of "vector bicouple" which has the components:

$$B_x = r\int_d 2\sigma_x N d\theta; \quad B_y = r\int_d 2\sigma_y N d\theta; \qquad B_z = r\int_d 2\sigma_z N d\theta. \tag{29}$$

This vector is defined when normal stress flows are known. In the case of a closed cone, the vector has two components only, and is perpendicular to the axis of the cone.

This concept of bicouple is an extension to cones of a similar concept previously introduced by the writer in the case of cylinders.[3] For cylinders, the bicouple is a scalar and has the dimension of a force multiplied by an area; for cones, the bicouple is a vector and has the dimension of a force. The difference is due to the fact that the dimensionless quantity σ_x is replaced by an area in the case of a cylinder. With notations (29), equations of the type of Eq. (21) or Eqs. (28) can be written in the form

$$\frac{M_x}{r_a} + B_x(r_a) = \frac{M_x}{r_b} + B_x(r_b). \tag{30}$$

This relation is of the same type as the relation between shearing forces and couples in the case of a cylindrical beam, the shearing forces being replaced by couples M_x, the couples being replaced by bicouples, and the arms being replaced by differences of reciprocals of radii. There are two relations such as (30) for closed cones, and three for open cones; to these conditions must be added condition (13), which expresses that the conical resultant is constant. In addition, the three conditions such as (19) can be written in the form

$$r_b(P_x(N_b) - P_x) = r_a(P_x(N_a) - P_x). \tag{31}$$

In these expressions, $P_x(N)$, $P_y(N)$, $P_z(N)$ are the components of the resultant $P(N)$ of the normal stress flows N. If the resultant $P(N)$ coincides with the true resultant P for one section, then, from Eqs. (31), this resultant P is constant and equal to P at each section.

Condition 4.—The shell is limited to two directrixes, which can redistribute shear, the loads to be transmitted by the shell and the normal stress flows along one directrix are known. There may be intermediate rigid frames.

The end loads N_a being known, together with components P_x, P_y, P_z, M_x, M_y, M_z, there exist 6 (closed shells) or 7 (open shells) conditions

which relate the end loads N_b to these quantities: the three conditions similar to Eq. (31), condition (13), and two conditions (closed shell) or three conditions (open shell) similar to Eq. (30). These conditions are generally insufficient to define the stresses, and the structure must be considered as statically indeterminate. This problem is examined in the next paragraph.

3. Solution of the Problem of Statically Indeterminate Conical Shells. Expression of the Conditions of Compatibility of Deformations by Considerations of Minimum Strain Energy

The general solution of the problem is highly intricate, especially if the frames are not considered as infinitely rigid. Even in the simpler case of cylinders, general solutions have not been entirely developed. An asymptotic solution may be obtained, which is similar to Saint Venant's solution for beams. This asymptotic solution can be used to define the distribution in a region of the shell which is separated by several frames from the loaded edge. It defines a state of stress in which sections are not distorted, which means that there is no tangential displacement along the line of intersection between the membrane and the frame. When this distribution is reached, the following frames are not loaded, and the same occurs when the applied loads follow this asymptotic distribution. In this case, there are no distorted sections and no frames are required (except to ensure the stability of the structure).

It is known that the corresponding beam distributions are independent of the magnitude of the shearing force, and are functions only of the applied axial load, bending couples,—and of bicouples in the case of open shells.[3] The property that shearing deformations do not affect distributions will be extended to the case of cones. It has been shown (Eqs. (7) and (8)) that the stress distribution on a conical shell can be represented by two groups of solution.

The first group corresponds to zero shear; this proves that for loads which can be in equilibrium with stresses of the first group, the asymptotic distribution is not a function of shear deformation.

In order to establish this property for the second group, consider a distribution defined by Eqs. (8), which has been obtained by application of the principle of minimum strain energy applied to normal stresses exclusively. Consider three consecutive frames, designated by a, b, c, respectively. On frames a and c, it is assumed that the normal stress flows are defined by Eq. (8). It will be shown that on frame b, the solution which makes the strain energy a minimum (including shear deformations), is also defined by Eq. (8). Assume that between frames a and b and between frames b and c_i respectively, the stress distributions are of the general type

$$\left.\begin{aligned} &\text{between } a \text{ and } b \\ &N=\frac{\phi'}{r^2}+\frac{\psi_a}{r}+\frac{\epsilon_a'}{r^2}; \quad T=\frac{\epsilon_a}{r^2}+\frac{\phi}{r^2}, \\ &\text{between } b \text{ and } c \\ &N=\frac{\phi'}{r^2}+\frac{\psi_c}{r}+\frac{\epsilon_c'}{r^2}; \quad T=\frac{\epsilon_c}{r^2}+\frac{\phi}{r^2}. \end{aligned}\right\} \tag{32}$$

In these expressions, ψ_a, ψ_c, ϵ_a, ϵ_c are arbitrary functions of θ, which satisfy the conditions of continuity of normal stress flow at points a, b, c, respectively

$$\psi_a/r_a+\epsilon_a'/r_a^2=0; \quad \psi_c/r_c+\epsilon_c'/r_c^2=0;$$
$$\psi_a/r_b+\epsilon_a'/r_b=\psi_c/r_c+\epsilon_c'/r_c.$$

Elimination of ψ_a and ψ_c leads to a relation between ϵ_a' and ϵ_b', which can be replaced in terms of a new function $\epsilon(\theta)$.

$$\epsilon_a'(1-r_b/r_a)=\epsilon_c'(1-r_b/r_c)=\epsilon'. \tag{33}$$

Then the total shear flow can be expressed as follows

$$\left.\begin{aligned} &\text{between } a \text{ and } b \quad T=\frac{\epsilon}{r^2}\frac{r_a}{r_a-r_b}+\frac{\phi}{r^2}, \\ &\text{between } b \text{ and } c \quad T=\frac{\epsilon}{r^2}\frac{r_c}{r_c-r_b}+\frac{\phi}{r^2}. \end{aligned}\right\} \tag{34}$$

If the thickness of the shell is constant along the generators, the strain energy in shear which corresponds to an infinitesimal interval between two generators is proportional to:

$$\int_{r_a}^{r_b}\frac{1}{r^2}\left[\phi+\frac{\epsilon r_a}{r_a-r_b}\right]^2 dr+\int_{r_b}^{r_c}\frac{1}{r^2}\left[\phi+\frac{\epsilon r_c}{r_c-r_b}\right]^2 dr.$$

This quadratic function of ϵ contains only terms independent of ϵ and terms proportional to ϵ^2 and is minimum when ϵ is equal to zero. This proves that the strain energy due to shear does not

affect the stress distribution in the case when the thickness of the skin resisting to shear is constant along each generator.

4. Determination of Stress Distributions which Satisfy the Conditions of Spatial Equilibrium and the Conditions of Compatibility of Deformations

The resultant and the resulting moment of the applied loads are known, but their distribution is unknown, so that the conditions of membrane equilibrium do not exist. Equations (7) and (8) define two groups of solutions: (*a*) Group I—transmission of a resultant passing through the apex (stress flows inversely proportional to the distance to the apex); (*b*) Group II—transmission of a couple (stress flows inversely proportional to the square of the distance to the apex).

(*a*) *Transmission of a Resultant Passing Through the Apex*

There is no shear, and Eqs. (15) can be written in the form

$$\int_d N\alpha_x d\theta = P_x/r. \tag{35}$$

Since all stress are inversely proportional to the radius, the condition of minimum strain energy can be reduced to

$$\int_d \frac{N^2 d\theta}{h} = \text{minimum}, \tag{36}$$

in which the integration in respect to r can be omitted. h is the thickness of the shell, constant along a generator, which may be variable along the directrix (following proportional laws of variation along all the generators).

Conventional methods of calculus of variations, or a beam analogy, show that the stress flows are defined by the relation

$$N = -\frac{h}{r}\sum k_{ij}\alpha_i P_j, \tag{37}$$

in which k_{ij} are coefficients defined by the substitution of expression (37) for N in Eq. (35), and i, j are indices which can be x, y, z. Substitution of notations

$$\int_d (\alpha_i)^2 h d\theta = I_{ii}; \quad \int_d \alpha_i \alpha_j h d\theta = I_{ij}, \tag{38}$$

into Eqs. (35), in which N is replaced by its expression (37), leads to nine equations of the two types

$$\left.\begin{aligned} k_{ii}I_{ii}+k_{ij}I_{ij}+k_{ig}I_{ig}&=1,\\ k_{ij}I_{ii}+k_{jj}I_{ij}+k_{jg}I_{ig}&=0. \end{aligned}\right\} \tag{39}$$

These equations are equivalent to six independent equations. The problem is greatly simplified if conventional methods of orthogonalization are applied. This leads to the determination of axes in reference to which the quantities I_{ij} are equal to zero. Then, expression (37) has the form.

$$N = -\frac{h}{r}\left(\frac{P_x\alpha_x}{I_{xx}}+\frac{P_y\alpha_y}{I_{yy}}+\frac{P_z\alpha_z}{I_{zz}}\right). \tag{40}$$

The orthogonalization thus defined is equivalent to the determination of the principal axes of inertia for the solid which coincides with the skew directrix defined by the coordinates $(\alpha_x, \alpha_y, \alpha_z)$, the weight of the curve per unit length being proportional to h; these principal directions will be called the principal directions of the directrix.

b. Transmission of a Couple

(b_1) *Transmission of a couple in an open conical shell.*—Stresses which are in equilibrium with a couple are defined by distributions of Group II (Eqs. (8)). They must satisfy conditions such as Eq. (16) which are equivalent to conditions such as Eq. (18). An integration by parts effected on Eq. (18) defines the relation

$$\int 2\sigma_x \phi' d\theta = -M_x, \tag{41}$$

which replaces Eq. (21). Two similar relations, corresponding to M_y and M_z can be written. The solution of these equations which satisfies the condition of minimum normal strain energy (it has been seen that the strain energy due to shear must be omitted) is of the type

$$\phi' = h\Sigma m_{ij} 2\sigma_i M_j, \tag{42}$$

or

$$N = \frac{h}{r^2}\sum m_{ij} 2\sigma_i M_j, \tag{43}$$

in which m_{ij} are coefficients defined by the substitution of expression (42) for ϕ' in equations such as (41), and the origins of the areas σ_i, indetermined until now, are defined by the conditions that the function ϕ be equal to zero at both edge

generators

$$\int_d h\sigma_x d\theta=0; \quad \int_d h\sigma_y d\theta=0; \quad \int_d h\sigma_z d\theta=0. \quad (44)$$

Introduction of the notations:

$$\int_d (2\sigma_i)^2 h d\theta=J_{ii}; \quad \int_d (2\sigma_i)(2\sigma_j) h d\theta=J_{ij}, \quad (45)$$

into conditions such as (41) in which ϕ' is replaced by its expression (42) leads to 9 equations of the two types

$$\left.\begin{aligned} m_{ii}J_{ii}+m_{ij}J_{ij}+m_{i9}J_{i9}&=1,\\ m_{ij}J_{ii}+m_{jj}J_{ij}+m_{j9}J_{j9}&=0. \end{aligned}\right\} (46)$$

In same manner as in the previous loading condition, an orthogonalization defined by conditions $J_{ij}=0$, can be effected and the solution (43) can be reduced to

$$N=\frac{h}{r^2}\left(\frac{M_x 2\sigma_x}{J_{xx}}+\frac{M_y 2\sigma_y}{J_{yy}}+\frac{M_z 2\sigma_z}{J_{zz}}\right). \quad (47)$$

The corresponding shear flow is

$$T=\frac{1}{r^2}\left(\frac{M_x}{J_{xx}}\int 2\sigma_z h d\theta+\frac{M_y}{J_{yy}}\int 2\sigma_y h d\theta+\frac{M_z}{J_{zz}}\int 2\sigma_z h d\theta\right). \quad (48)$$

This orthogonalization is equivalent to the determination of the principal axes of inertia of a solid line having elemental weights $h\delta\theta$ at points of coordinates $(\sigma_x, \sigma_y, \sigma_z)$; these principal directions will be called the principal directions of the membrane.

(b_2). *Transmission of a couple in a closed conical shell.*—It has already been shown that the direction called axis of the shell has the following property: if a couple is applied along this axis the transmission of stress is effected by shears exclusively.

The axis z coinciding with the axis of the cone, the normal stress distribution is defined by expressions of the type

$$N=\frac{h}{r^2}(m_{xx}2\sigma_x M_x+m_{xy}(2\sigma_y M_x+2\sigma_x M_y)+m_{yy}2\sigma_y M_y). \quad (49)$$

The principal axes of inertia in the plane (x, y) can be determined in the same manner as in the space. Introducing the notations

$$\int (2\sigma_x)^2 h d\theta=J_{xx}; \quad \int_d 4\sigma_x\sigma_y h d\theta=J_{xy}=0; \quad \int_d (2\sigma_y)^2 h d\theta=J_{yy}, \quad (50)$$

the distribution is defined by

$$N=\frac{h}{r^2}\left[\frac{2\sigma_x M_x}{J_{xx}}+\frac{2\sigma_y M_y}{J_{yy}}\right];$$

$$T=\frac{1}{r^2}\left(\frac{M_x}{J_{xx}}\int 2\sigma_x h d\theta+\frac{M_y}{J_{yy}}\int 2\sigma_y h d\theta\right)+\frac{M_z}{2r^2\Sigma}. \quad (51)$$

The quantity Σ in the second equation is the area defined in Eqs. (24). The origins of the two integrals are defined by the conditions that the corresponding shears T do not have a resulting couple M_z.

5. Determination of Stress Distributions Which are in Membrane Equilibrium with the Normal Loads Applied along a Directrix and Which Satisfy the Conditions of Spatial Equilibrium and of Compatibility of Deformations

This problem differs from that of the preceding paragraph by the fact that not only is the resultant of applied loads known in magnitude and position, but also the mode of application of this resultant is determined. The solutions obtained in the previous paragraph are inadequate because they satisfy the conditions of spatial equilibrium only, while the conditions of membrane equilibrium are disregarded. The "lack of equilibrium" can be determined as follows.

Let: N_a the applied normal stress flows at the radius $r=r_a$, N_a', T_a', the values of the stress flows, calculated by means of the formulas of the preceding paragraph, for the same radius r_a ("basic distribution").

$\Delta N_a=N_a-N_a'$, residual stress flows. The residual stress flows ΔN_a are in spatial equilibrium with the difference between the resultant (or resulting couple) of the shear flows T_a' and the force (or couple) applied to the frame. These residual stresses have a conical resultant R and a resulting bicouple B.

(a) *The Conical Resultant R and the Bicouple B are Both Equal to Zero.*

The applied loads are in membrane equilibrium with the basic stress flows. The transition between the applied loads and the basic distribution may be practically effected in a region limited by two redistributing frames which separate the applied loads from the region where the stresses are assumed to follow the basic distribution.

(b) *The Conical Resultant R or the Bicouple B, or Both, Are Different from Zero*

The conical resultant R is constant along the cone. Since the couples have been balanced, equations of the type of (30) are reduced to the condition of constant bicouple. Both quantities R and B are forces and are independent of the radius, and the corresponding stress distributions are defined by solutions of the Group I (Eqs. (7)); they are of the type

$$\Delta N=\Delta\psi/r. \tag{52}$$

They must satisfy the following conditions:

$$\text{One condition} \qquad \int_d \Delta\psi d\theta=\Delta R. \tag{53}$$

$$\left.\begin{array}{l}\text{Two conditions}\\ \text{(closed cone)}\\ \text{Three conditions}\\ \text{(open cone)}\end{array}\right\} \int_d \Delta\psi 2\sigma_i d\theta=\Delta B_i. \tag{54}$$

In Eq. (54), index i stands for x,y (closed cone) or for x,y,z (open cone). In addition, the three conditions of zero spatial resultant and the condition of minimum strain energy must be satisfied.

$$\int_d \Delta\psi\alpha_i d\theta=0, \tag{55}$$

$$\int_d (\Delta\psi)^2\frac{d\theta}{h}=\text{minimum}. \tag{56}$$

The solution of the system of Eqs. (53)–(56) can be expressed in the form

$$\frac{\Delta\psi}{h}=\Sigma\Sigma\lambda_{ij}\alpha_i\Delta B_j+\Sigma\Sigma\mu_{ij}2\sigma_i\Delta B_j+\Sigma\nu_i\Delta B_i + \Sigma\eta_i\alpha_i\Delta R+\Sigma\nu_i 2\sigma_i\Delta R+\Delta R/a_0. \tag{57}$$

λ_{ij}, μ_{ij}, ν_i, η_i, a_0 are coefficients which are defined by the linear equations obtained by substitution of expression (57) to $\Delta\psi/h$ in conditions (53)–(55). This defines 28 equations; 22 of them are independent, a number equal to the number of unknowns (it must be noted that $\lambda_{ij}\neq\lambda_{ji}$, while $\mu_{ij}=\mu_{ji}$).

Methods of orthogonalization can be applied; in view of the great number of independent unknowns, they will be advantageous only if many different loading conditions are examined. They can be applied to the total stresses N directly, without introduction of the residual stresses. Then, calculating the values of the resultant, the resulting bicouple and the conical resultant of the applied loads, Eqs. (53)–(55) become

$$N=\psi/r, \tag{58}$$

$$\int_d \psi d\theta=R; \quad \int_d \psi\alpha_i d\theta=P_i; \quad \int_d \psi 2\sigma_i d\theta=B_i. \tag{59}$$

The solution of these equations which satisfies the condition of minimum energy is

$$N=\frac{\psi}{r}=\frac{h}{r}[\Sigma\Sigma\chi_{ij}\alpha_i P_j+\Sigma\Sigma\lambda_{ij}(\alpha_i B_j+2\sigma_i P_j) + \Sigma\Sigma\mu_{ij}2\sigma_i B_j+\Sigma\eta_i R+\Sigma\nu_i B_i + \Sigma\eta_i\alpha_i R+\Sigma\nu_i 2\sigma_i R+R/a_0]. \tag{60}$$

Orthogonalization can be effected by Gauss' method, which can be adapted to make use of the symmetry of the problem. The coefficient a_0 is calculated first, using first Eq. (59)

$$a_0=\int_d h d\theta. \tag{61}$$

Then, a linear transformation is effected on all quantites α_i, $2\sigma_i$

$$\alpha_i'=\alpha_i-\bar{\alpha}_i; \quad \sigma_i'=\sigma_i-\bar{\sigma}_i, \tag{62}$$

the constants $\bar{\alpha}_i$, $\bar{\sigma}_i$ being chosen in such manner that coefficients ν_i, η_i be equal to zero,

$$\bar{\alpha}_i=\int \alpha_i h d\theta \Big/ a_0; \quad \bar{\sigma}_i=\int \sigma_i h d\theta \Big/ a_0. \tag{63}$$

The quantities α_i' can be orthogonalized between themselves, by a rotation of axes. Calling β_i the new quantities α_i, orthogonalization is effected between quantities σ_i and β_i; with

$$\sigma_i''=\sigma_i'+\Sigma\xi_{ik}\beta_k, \tag{64}$$

the conditions of orthogonalization are

$$\int_d \sigma_i'' \beta_\epsilon h d\theta = \int_d \sigma_i' \beta_\epsilon h d\theta + \xi_{i\epsilon} \int_d (\beta_\epsilon^*)^2 h d\theta = 0, \quad (65)$$

which defines coefficients $\xi_{i\epsilon}$; it can be noted that

$$\left.\begin{aligned} \int_d (\sigma_i'')^2 h d\theta &= \int_d (\sigma_i')^2 h d\theta - \Sigma \xi_{i\epsilon}^2 \int_d (\beta_\epsilon)^2 h d\theta, \\ \int_d \sigma_i'' \sigma_j'' h d\theta &= \int_d \sigma_i' \sigma_j' h d\theta \\ &\quad - \Sigma \xi_{i\epsilon} \xi_{j\epsilon} \int_d (\beta_\epsilon)^2 h d\theta. \end{aligned}\right\} (66)$$

The final orthogonalization is effected between quantities σ_i'', by a rotation of axes which defines quantities γ_i linearly related to the quantities σ_i''.

The orthogonalization being effected, the expression of N is

$$N = \frac{\psi}{r} = \frac{h}{r}\left[\Sigma \frac{P_i^* \beta_i}{\int_d (\beta_i)^2 h d\theta} + \Sigma \frac{B_i^* 2\gamma_i}{\int_d (2\gamma_i)^2 h d\theta} + \frac{R}{a_0}\right]. \quad (67)$$

The new quantities P_i^*, B_i^* result from P_i, B_i by the same linear transformation as quantities β_i, γ_i result from α_i, σ_i; quantities P_i^* are linear functions of P_i and R, quantities B_i^* are linear functions of B_i, P_i and R. Equation (60), or the simplified Eq. (67) must replace Eq. (40) when the distribution of the applied loads is known. All these equations are somewhat simplified for a closed cone, since the bicouple has only two components in this case. It can be noted that only the terms corresponding to the first group of solution (ψ/r) are functions of the load distribution; the terms corresponding to the second group of solutions (ϕ/r^2) are independent of the load distribution.

III. CONICAL SHELLS WITH PLANE FRAMES

It has been indicated that the case of cones in the shape of surfaces of revolution is much less complex than the general case, because the number of conditions of equilibrium is reduced by three units (see Table II). The simplification subsists in the case of general cones with plane frames, but the method of analysis developed hereafter is theoretically incorrect in the case of statically indeterminate structures, which corresponds to all problems related to the determination of stress distributions. However, it is possible to use the simplified method and avoid the complications of the exact method if the angles at the apex of the conical surface with the normal to the plane directrix are relatively small or vary between narrow limits; in the first case the structure is similar to a cylinder (Fig. 5b), in the second, it is similar to a cone of revolution (Fig. 5a).

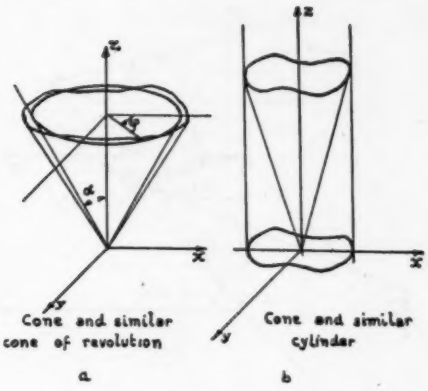

FIG. 5.

The solutions of the problem of stress distributions defined hereafter are theoretically incorrect, except for these two limit cases of cylinders and cones of revolution. A justification of the necessity of the modification of the fundamental analysis resides in the fact that this analysis cannot account for the circumstance that the frames being plane structures, the stress transmission at the frames does not follow exactly the laws assumed in the analysis. In addition, since loads are generally distributed at all the frames, it results that the theoretical distributions found heretofore are not actual distributions, but theoretical limits; also, variations of thickness along the generators have the same effects as distributed loads at the frames. The shape of the frames indicates the simplified method to be used. Instead of considering true normal and shearing stresses, oblique components are used. The two directions of the shear stress at each point are not orthogonal, and coincide with the generator and the parallel to the frames which is in the plane tangent to the cone. The direction of the axial stress coincides with the generator. It has been shown[1] that these directions do not define conventional stresses in rectangular coordinates, so that the relations between stresses and strains should be corrected. In order to solve statically indeterminate problems, it is assumed

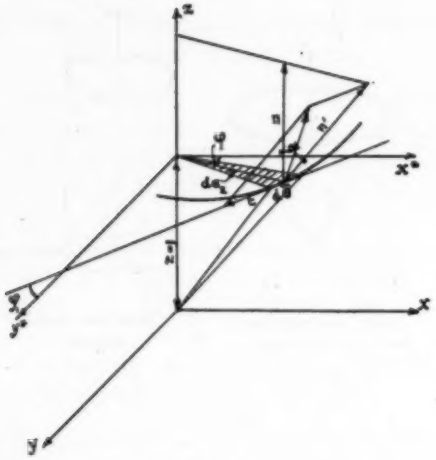

FIG. 6.

that the stress strain relations which are valid in rectangular axes can be applied to slightly oblique directions without important errors. This implies that the angles between the generators and the plane directrixes are but slightly different from right angles, which is true in the cases indicated above. In these cases, conventional methods of strain energy can be applied to the axial stresses as if they were true normal stresses. In this analysis, stress flow components are defined as follows (Fig. 6): the stress flow acting on the unit length of plane directrix is decomposed into a component t parallel to the directrix (shear flow) and a component n' along the generators; n' can be projected on the axis of the directrix (perpendicular to the plane of the directrix) and defines the reduced stress flow $n=n'\cos\alpha$, α being the angle between the generator and the axis of the directrix. It can be shown[1] that $t=T$ when there are no surface loads. In the case of statically determinate problems, the use of the components n instead of N is entirely legitimate, provided the equations of equilibrium be correctly written.

In the case of statically indeterminate problems, it is assumed that the component n' defines the axial strain. Since this component changes with the direction of the axis of the directrix, it appears that the solutions defined hereafter are modified when the direction of the axis is changed. Provided the variations remain within close limits, the method can be used for practical applications.

1. Conditions of Membrane Equilibrium

The independent variables which define a point on the surface of the cone are z, coordinate along the axis of the directrix, and $\bar{\theta}$, length of arc along the directrix defined by $z=1$ (Fig. 7).

The equation of equilibrium between reduced stress flows n and t is obtained by projecting on the z axis all the stresses acting on an infinitesimal element $(dz, d\theta)$

$$t\,dz-\left(t+\frac{\partial t}{\partial\bar{\theta}}d\bar{\theta}\right)dz+n(z\,d\bar{\theta})-\left(nz+\frac{\partial(nz)}{\partial z}dz\right)d\bar{\theta}=0.$$

Simplifying this expression, it is found

$$\partial t/\partial\bar{\theta}+\partial(nz)/\partial z=0. \tag{68}$$

It has been previously shown that the shear flow $t=T$ is inversely proportional to r^2 (along a generator). Since r and z are proportional along a generator, it results that

$$t=\phi(\bar{\theta})/z^2. \tag{69}$$

In this expression, the stress function ϕ is different from that used previously, the ratio between the two being equal to $1/\cos^2\alpha$. Substitution of expression (68) into Eq. (69) leads to the general solution for the reduced normal stress flow

$$n=\phi'(\bar{\theta})/z^2+\psi(\bar{\theta})/z. \tag{70}$$

This shows that the two groups of solutions defined in the case of spherical directrixes subsist in the present case (see Eqs. (7) and (8)).

2. Statically Determinate Problems

Condition 1.—The reduced stress flows n_a and t_a are known along a directrix z_a. The solution is similar to that given by Eqs. (9):

$$\phi=t_a z_a^2;\quad \psi=n_a z_a-\phi'/z_a. \tag{71}$$

Condition 2.—The reduced stress flows n_a, n_b along two fixed directrixes (z_a, z_b) are known.

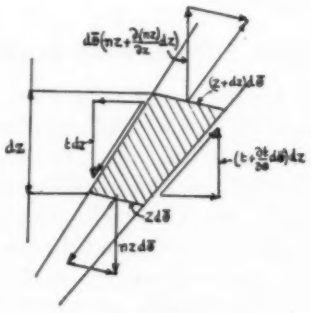

FIG. 7.

The solution is similar to that given by Eqs. (10)–(12)

$$\psi=\frac{n_a z_a^2-n_b z_b^2}{z_a-z_b};\quad \phi'=\frac{n_a z_a-n_b z_b}{(1/z_a)-(1/z_b)}. \tag{72}$$

Condition (13) becomes

$$\int_d n_b(z_b d\bar{\theta})=\int_d n_a(z_a d\bar{\theta})=P_z, \tag{73}$$

in which P_z is the load component normal to the directrix. Thus, the special condition of conical resultant (13) is eliminated when plane frames are used, and is replaced by the condition of spatial equilibrium, valid for both closed and open shells.

Condition 3.—The reduced stress flows n_a, n_b along two directrixes (z_a, z_b) and all the load and moment components are known. The relationship between the component P_z and the stress flows is given by Eq. (73), which must be satisfied.

The two equations of equilibrium along axes x and y are similar to Eqs. (15) and can be transformed in similar manner. It is thus found, instead of Eqs. (31), two equations similar to

$$M_y(b)+P_x z_b=M_y(a)+P_x z_a, \tag{74}$$

in which (Fig. 6)

$$M_y(a)=-\int_d x^* n(z_a^2 d\bar{\theta}), \tag{75}$$

x^* and y^* being the coordinates of the unit directrix ($z=1$), and $M_y(a)$ being the moment around the y^* axis due to the normal components n of the axial stresses.

The three equations of equilibrium of moments are

$$\int_d tz^2 dx^*=M_y;\quad \int_d tz^2 dy^*=-M_x; \tag{76}$$

$$2\int_d tz^2 d\sigma=M_z. \tag{77}$$

In these equations, dx^*, dy^* replace $-d\sigma_y$ and $d\sigma_x$ of Section II. Integrals $\int_d dx^*$ and $\int_d dy^*$ are equal to zero for a closed cone, a fact which corresponds to the remark that the z axis is the axis of torque for a cone limited by plane directrixes.

The previous equations can be integrated by parts, using the second Eq. (72) to define $t'z^2=\phi'$. Equations similar to (21), (27), (28) are obtained. Because of the preceding remark, the equations containing M_x and M_y are the same for both cases of open and closed cones and are of the type

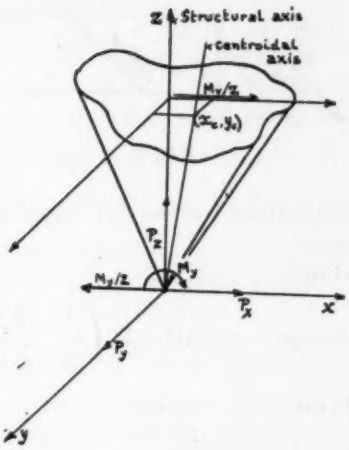

FIG. 8.

$$\int_d (z_a n_a-z_b n_b)x^* d\bar{\theta}=\frac{M_y(b)}{z_b}-\frac{M_y(a)}{z_a}=M_y\left(\frac{1}{z_b}-\frac{1}{z_a}\right). \tag{78}$$

Solving Eqs. (74) and (78), it is found

$$M_y(a)=M_y-z_a P_x;\quad M_y(b)=M_y+z_b P_x, \tag{79a}$$

and, similarly

$$M_x(a)=M_x+z_a P_y;\quad M_x(b)=M_x+z_b P_y. \tag{79b}$$

These Eqs. (79) have a very simple significance and can also be established as follows (Fig. 8): An applied force P_x (at the apex) is in equilibrium with axial stresses along the generators, the normal components of which at a section "a" parallel to P_x constitute a couple equal to $-P_x z_a$, and the in-plane components of which have a resultant equal to P_x. Similarly, an applied couple M_y is in equilibrium with the normal components of the axial stresses, while the in-plane components are in equilibrium with the shear flows, each of the two partial resultants being equal to $\pm M_y/z$. Thus, in the general case of combined load at the apex and couple, the moment $M_y(a)$ of the normal components of the axial stresses is equal to M_y-P_x. This establishes Eqs. (79a), (79b) in a direct manner, and also shows why similar relations cannot be established with spherical directrixes, since it is impossible to define a moment of normal components.

Integration by parts of Eq. (76) leads to two

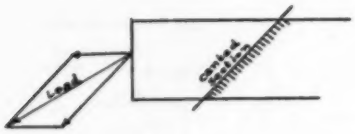

FIG. 9.

different equations for the two cases of open and closed cones:

Open cones:

$$\int_d 2\sigma(z_a n_a - z_b n_b)d\tilde{\theta} = M_z\left(\frac{1}{z_b} - \frac{1}{z_a}\right). \qquad (80)$$

Closed cones:

$$\left(\frac{1}{z_a} - \frac{1}{z_b}\right)C\Sigma - \int_d 2\sigma(z_a n_a - z_b n_b)d\tilde{\theta} = M_z\left(\frac{1}{z_b} - \frac{1}{z_a}\right). \qquad (81)$$

Σ is the area enclosed by the unit directrix, and C is a constant of integration which is defined by Eq. (81). This constant defines the value of ϕ (resulting from integration of ϕ'—Eq. (72)), at the point corresponding to $\sigma = 0$. Thus, in the case of closed cones with plane directrixes, there are no special equations of equilibrium for membranes. On the contrary, Eq. (80) which applies to open cones is a special condition of equilibrium for membranes, which must be added to the spatial conditions of equilibrium, Eqs. (73), (79a), (79b) which are equivalent to the condition of equilibrium of forces and the two conditions of equilibrium of moments for beams. Equation (80) defines bicouple which is similar to bicouples defined by Eqs. (29)

$$B = \int 2\sigma n z d\tilde{\theta}. \qquad (82)$$

This quantity can be considered as a scalar, or as a vector directed along the axis z, which coincides with the definition (29) when the components B_x, B_y are disregarded. When considered as a scalar, this concept is similar to that of a bicouple previously introduced in the case of cylinders, with the differences indicated above. The relation between couples and bicouples is

$$M_z/z_a + B(z_a) = M_z/z_b + B(z_b). \qquad (83)$$

3. Statically Indeterminate Problems

The results of Section II, 3 can be directly applied to a cone of revolution, and plane sections of reference arc perpendicular to the axis, which is known in this case. In the general case, plane sections are defined by the existing frames, which means, as noted above, that the stress distribution is a function of the frame direction. This result is obvious for cylinders with oblique frames (Fig. 9): it is not possible to calculate stresses by means of "corrected oblique section properties," in conjunction with some appropriate method of load decomposition; successive straight sections must be used, although some contradictions arise at the loading frames, which are neglected in conventional stress analysis. The method can be extended to cones, but no "correct" axis can be defined, because "correct" sections are spherical. To some extent, due to the approximations made, the direction of the axis is somewhat a question of personal judgment. However, some general considerations, together with requirements of simplicity of calculation, may lead to specific solutions.

(a) *Definition of Plane Sections*

In the case of closed cones, the axis of torque may often be used as direction of the axis perpendicular to plane sections, which will be called "structural axis." Because of this simplification, the determination of two distinct axes is avoided, but the axis of torque may be unsatisfactory from the standpoint of the following considerations. From Eq. (17) of reference 1, the relation between the true normal stress flow N and the component n' is

$$n' = N \sin u + 2T \cos u, \qquad (84)$$

in which u is the angle between the generator and the plane directrix intersecting at a point of the cone. If u is approximately equal to a right angle, n' and N are practically equal, and the error resulting from the confusion between n' and N in the expressions of strain energy may be negligible; u is not very different from a right angle if the angle of the generators with the structural axis varies between narrow limits since, taking the structural axis as axis z and considering the unit directrix ($z = 1$)

$$\cos u = \frac{x^* dx^* + y^* dy^*}{(1 + x^{*2} + y^{*2})^{\frac{1}{2}}(dx^{*2} + dy^{*2})^{\frac{1}{2}}},$$

$$\cos \alpha = \frac{1}{(1 + x^{*2} + y^{*2})^{\frac{1}{2}}}; \quad d\tilde{\theta} = (dx^{*2} + dy^{*2})^{\frac{1}{2}};$$

so that

$$\cos u=(d/d\bar{\theta})(1/\cos\alpha). \tag{85}$$

When α is constant (cone of revolution) and especially when $\alpha=0$ (cylinder), it appears that u is a right angle. Thus, the structural axis must be chosen in such manner that α be as constant as possible; this condition can be written by expressing that the quadratic dispersion of the values of $\cos\alpha$ must be minimum. Since loads along generators are proportional to areas $hd\bar{\theta}$ it is logical to give weights $hd\bar{\theta}=da$ (elemental areas) to the quantities $\cos\alpha$. Then, the condition of minimum of the quadratic dispersion is

$$\delta\int_d(\cos\alpha-\cos\alpha_0)^2da=0, \tag{86}$$

in which $\cos\alpha_0$ is an average value of $\cos\alpha$. ϵ_x, ϵ_y, ϵ_z being the direction cosines of the structural axis, and α_x, α_y, α_z the direction cosines of a generator, the following relations exist

$$\begin{gathered}\cos\alpha=\epsilon_x\alpha_x+\epsilon_y\alpha_y+\epsilon_z\alpha_z;\\ \cos\alpha_0=\int_d\frac{\cos\alpha da}{a_0};\quad a_0=\int_d da.\end{gathered} \tag{87}$$

Equation (86) can be written as

$$\delta W=\delta\left[a_0\int_d\cos^2\alpha da-\left(\int_d\cos\alpha da\right)^2\right]=0, \tag{88}$$

equivalent to

$$\frac{\partial W}{\partial\epsilon_x}\delta\epsilon_x+\frac{\partial W}{\partial\epsilon_y}\delta\epsilon_y+\frac{\partial W}{\partial\epsilon_z}\delta\epsilon_z=0, \tag{89}$$

with

$$\frac{\partial W}{\partial\epsilon_x}=a_0\int_d\cos\alpha\alpha_x da-\int_d\cos\alpha da\int\alpha_x da, \tag{90}$$

and, since ϵ_x, ϵ_y, ϵ_z are direction cosines,

$$\epsilon_x\delta\epsilon_x+\epsilon_y\delta\epsilon_y+\epsilon_z\delta\epsilon_z=0. \tag{91}$$

Conditions (89), (91) are simultaneously satisfied if

$$\frac{1}{\epsilon_x}\frac{\partial W}{\partial\epsilon_x}=\frac{1}{\epsilon_y}\frac{\partial W}{\partial\epsilon_y}=\frac{1}{\epsilon_z}\frac{\partial W}{\partial\epsilon_z}=S. \tag{92}$$

These relations lead to the well known equation in S

$$\begin{vmatrix}A_{xx}-S & A_{xy} & A_{xz}\\ A_{xy} & A_{yy}-S & A_{yz}\\ A_{xz} & A_{yz} & A_{zz}-S\end{vmatrix}=0. \tag{93}$$

In this expression (see Eqs. (38), (63))

$$A_{xx}=I_{xx}-a_0\bar{\alpha}_x^2,\quad A_{xy}=I_{xy}-a_0\bar{\alpha}_x\bar{\alpha}_y. \tag{94}$$

There are three solutions for S, corresponding to three perpendicular directions defined by three series of Eqs. (92), which are

$$(A_{xx}-S)\epsilon_x+A_{xy}\epsilon_y+A_{xz}\epsilon_z=0, \tag{95}$$

and two similar equations. One of the three solutions defines the structural axis, and the two other solutions can be used to constitute with the first a system of new tri-rectangular axes.

The sequence of operations which lead to the structural axis is indicated on Table III. In this table, it is assumed that the cone is defined by an arbitrary plane directrix, at a distance $z=1$ from the apex.

The axis of torque can be determined in the manner indicated in Section II, 2, condition 3 (Eqs. (24)). Generally, the initial axis z is close to the axis of torque, and Eqs. (24) are simplified:

$$\Sigma=\Sigma_z;\quad \omega_x=\Sigma_x/\Sigma_z;\quad \omega_y=\Sigma_y/\Sigma;\quad \omega_z=1. \tag{96}$$

Table IV shows the method of analysis in the general case, without the last simplification. When the axis of torque (z_1) is defined, arbitrary rectangular axes x_1, y_1 are chosen (they must be perpendicular to z_1). In Table IV, the axis x_1 is *chosen* in the plane (x, y), which means that it is perpendicular to the projection of z_1 to this plane; z_1 being defined by the direction cosines $(\omega_x, \omega_y, \omega_z)$, the direction x_1 is defined by $(\alpha_x, \alpha_y, 0)$ such as $\alpha_x\omega_x+\alpha_y\omega_y=0$. When the axis x_1 is chosen, the axis y_1 is entirely determined; its direction cosines are $(\beta_x, \beta_y, \beta_z)$, defined as shown. The areas limited by a spherical directrix are defined by expressions such as

$$\Delta\sigma_z=y_n'z_{n+1}'-y_{n+1}'z_n',$$

in which the index (′) represents points on the unit spherical directrix. Introducing the radius $r=(1+x^2+y^2)^{\frac{1}{2}}$ it is found: $x'=x/r$; $y'=y/r$; $z'=1/r$ and

$$\left.\begin{aligned}\Delta\sigma_x&=-(y_{n+1}-y_n)/r_nr_{n+1},\\ \Delta\sigma_y&=(x_{n+1}-x_n)/r_nr_{n+1},\\ \Delta\sigma_z&=(x_ny_{n+1}-x_{n+1}y_n)/r_nr_{n+1}.\end{aligned}\right\} \tag{97}$$

The structural axis being determined, either

TABLE III. Structural axis in a cone.

Point	x^*	y^*	a	a_x	a_x	a_y	aa_x	aa_y	aa_z	I	A, S	ϵ	α_x'	α_y'	α_z'	x'	y'
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1																	
2																	
.																	
.																	
.																	
n																	
.																	
.																	
.																	
p																	
Totals			Σa				Σaa_x	Σaa_y	Σaa_z								

[1] Reference number of a generator.
[2, 3] Coordinates of the generator ($z=1$).
[4] Elemental area.
[5–7] Direction cosines $\alpha_x=1/(1+(x^*)^2+(y^*)^2)^{1/2}$; $\alpha_x=\alpha_x x^*$; $\alpha_y=\alpha_x y^*$.
[8–10] Determination of $\bar{\alpha}_x$, $\bar{\alpha}_y$, $\bar{\alpha}_z$ (by summation), $\bar{\alpha}_x=\Sigma a\alpha_x/\Sigma a$; $\bar{\alpha}_y=\Sigma a\alpha_y/\Sigma a$; $\bar{\alpha}_z=\Sigma a\bar{\alpha}_z/\Sigma a$.
[11] Determination of I_{xx}, I_{xy}, I_{xz}, I_{yy}, I_{yz}, I_{zz}, by multiplication between columns (5)–(7) and (8)–(10) and progressive summation.
[12] Determination of $A_{xx}-A_{zz}$ by means of Eqs. (94), and of three roots S (S^x, S^y, S^z) by Eq. (93); S^z is the structural axis, chosen inside the cone.
[13] Determination of three sets of values of ϵ_x, ϵ_y, ϵ_z, corresponding to the three roots S, by Eqs. (95) and similar. To S^z correspond $\epsilon_x{}^z$, $\epsilon_y{}^z$, $\epsilon_z{}^z$.
[14–16] New direction cosines, such as $\alpha_x'=\alpha_x\epsilon_x{}^z+\alpha_y\epsilon_y{}^z+\alpha_z\epsilon_z{}^z$.
[17, 18] Coordinates for the new unit directrix, $x'=\alpha_x'/\alpha_z'$, $y'=\alpha_y'/\alpha_z'$.

"correctly" by the first method, or "approximately," by the second method, or less exactly by chosing a direction which seems to represent the structural axis, or even by using the direction normal to the frames if this direction seems acceptable, the stress distribution under given applied resulting loads can be examined.

(b) *Determination of Stress Distributions*

Stress distributions are determined by application of methods of strain energy, the normal strains being defined by the component n of the stress flow. When the shear flow is neglected, Eq. (84) becomes

$$n=N\sin u\cos\alpha. \tag{98}$$

The ratio between $rd\theta$ (along the spherical directrix) and ds (along the plane directrix) is $\sin u$; the elemental strain energy is thus

$$dW=\frac{N^2rd\theta dr}{2Eh}=\frac{n^2dsdz}{2Eh\sin u\cos^3\alpha}, \tag{99}$$

TABLE IV. Axis of torque in a closed cone.

Point	x^*	y^*	r	Δx^*	Δy^*	$x^*\Delta y^*-y^*\Delta x^*$	r_nr_{n+1}	$\Delta\sigma_x$	$\Delta\sigma_y$	$\Delta\sigma_z$	x_1'	y_1'	z_1'	x_1	y_1
1	2	3	5	6	7	8	9	10	11	12	13	14	15	16	17
1															
2															
.															
.															
.															
n															
.															
.															
.															
p															
1				—	—	——	——								
Totals								Σ_x	Σ_y	Σ_z					

[1] Reference number of a generator.
[2, 3] Coordinates of a generator ($z=1$). } See Table III.
[4] $r=1/\alpha_z=(1+(x^*)^2+(y^*)^2)^{1/2}$, length of a generator (to $z=1$).
[6, 7] Differences between consecutive values in columns (2), (3).
[8] Expression (2)(7)–(3)(6).
[9] Product of consecutive values in column (5).
[10–12] Elemental areas (see Eqs. 85): $\Delta\sigma_x=-\Delta y^*/r_nr_{n+1}$; $\Delta\sigma_y=\Delta x^*/r_nr_{n+1}$; $\Delta\sigma_z=(x^*\Delta y^*-y^*\Delta x^*)/r_nr_{n+1}$.
Total areas Σ_x, Σ_y, Σ_z obtained by summation.
Direction cosines of the axis of torque (axis z_1): $\omega_x=\Sigma_x/\Sigma$; $\omega_y=\Sigma_y/\Sigma$; $\omega_z=\Sigma_z/\Sigma$; $\Sigma=(\Sigma_x^2+\Sigma_y^2+\Sigma_z^2)^{1/2}$.
Direction cosines of the axis x_1: $\alpha_x=\sin\chi$; $\alpha_y=\cos\chi$; $\alpha_z=0$ with $\tan\chi=-\Sigma_y/\Sigma_x$.
Direction cosines of the axis y_1: $\beta_x=b_x/b$; $\beta_y=b_y/b$; $\beta_z=b_z/b$; with $b_x=\alpha_y\omega_z$; $b_y=-\alpha_x\omega_z$; $b_z=\alpha_x\omega_y-\alpha_y\omega_x$; $b=(b_x^2+b_y^2+b_z^2)^{1/2}$.
[13–15] Coordinates of the point $(x, y, 1)$ in the new axes: $x_1'=x\alpha_x+y\beta_x+\omega_x$; $y_1'=x\alpha_y+y\beta_y+\omega_y$; $z_1'=y\beta_z+\omega_z$.
[16, 17] Coordinates of the intersection of a generator with the plane $z_1=1$: $x_1=x_1'/z_1'$; $y_1=y_1'/z_1'$; $z_1=1$.

the apparent thickness is h'

$$h' = h \sin u \cos^3 \alpha. \tag{100}$$

If there are concentrated areas (stringers, longerons, etc.), the apparent areas a' are related to the actual areas a by

$$a' = a \cos^3 \alpha. \tag{100a}$$

Apparent thicknesses or areas define conventional section properties in the planes perpendicular to the structural axis. While total areas, moments of inertia and statical moments are calculated for each section, Eq. (82) shows that section properties related to bimoments must be calculated with the unit section polar areas σ (but with the actual developed lengths of directrix and actual thicknesses).

(b_1) *Normal stresses.*—The conventional formulas of the beam theory can be applied, because Eqs. (73) to (83) are similar to the conventional relations between normal stresses and axial loads, bending moments and bimoments, a small difference existing in the case of bimoments (see Eqs. (82), (83)).

It results from that remark that methods of strain energy will define the same normal stress distribution as in cylindrical beams, provided the bending moments be calculated in respect to the centroids of the sections, which do not coincide with the structural axis, at least in general, and the bimoments be calculated about the conventional shear centers. As indicated, unit polar areas, instead of actual areas, are used throughout in the determination of bimoments and corresponding section properties. This difference results from the fact that bicouples are forces instead of forces $\times$ areas, a fact which explains why Eq. (83) differs from the conventional relation: $B(b) - B(a) = (z_b - z_a) M_z$, valid for cylinders.

(b_2) *Shears.*—Shear flows can be defined by conventional expressions, provided the transversal forces be correctly defined. This leads to modify the applied transversal forces by the so called "taper effect." Obviously, there is no taper effect when torques only are applied: under the action of a torque M_z, there is a constant shear flow along a closed section, or a shear flow defined by the torsion bending theory for beams, along an open section.

In the general case, it has been noted that forces applied to the apex do not produce shears, the latter being due to the couples M_x, M_y, M_z calculated about the apex. The couples M_x, M_y can be replaced by force components $P_x(z)$, $P_y(z)$ which are equal to

$$\left.\begin{aligned} P_y(z) &= -\frac{M_x}{z} = P_y - \frac{M_x(z)}{z}, \\ P_x(z) &= \frac{M_y}{z} = P_x + \frac{M_y(z)}{z}, \end{aligned}\right\} \tag{101}$$

as results from Eqs. (79). The quantities $-M_y(z)/z$ and $+M_x(z)/z$ are the so-called "taper effects."

It must be noted that in Eqs. (101), the moments $M_x(z)$, $M_y(z)$ are calculated about the axes (x, y) passing in each section at the point of intersection with the structural axis, and not about centroidal axes.

If centroidal axes are considered, then Eqs. (101) must be modified in the case of an axial load P_z. They become, calling (x_c, y_c) the coordinates of the centroid in respect to the structural axes (Fig. 8), and introducing an axial force P_z positive when it is a compression (directed toward $z > 0$)

$$\left.\begin{aligned} P_y(z) &= P_y - \frac{M_x^c(z)}{z} - P_z \frac{y_c}{z}, \\ P_x(z) &= P_x + \frac{M_y^c(z)}{z} - P_z \frac{x_c}{z}. \end{aligned}\right\} \tag{102}$$

If the torque M_z is calculated in reference to the structural axis, it must not be corrected for taper effect, and the shear distribution is calculated under the action of $P_y(z)$, $P_x(z)$ and M_z by conventional methods applied to the cylindrical beam which has the local section properties of the cone.

4. Conclusions

A conical beam can be analyzed in the same manner as a cylindrical beam. Since there is no natural direction of plane sections, which are replaced by spherical sections, an approximately "correct" direction of plane sections must be defined; in Section (III, 3a), two methods have been developed, one defining the axis of torque, the other defining a special direction called "structural axis," to which "correct" sections are perpendicular. When correct sections are defined, conventional methods give fairly ac-

curate results,—as compared with those of the exact theory (Section II),—when the cone has a small angle at the apex, or when the cone is not very different from a cone of revolution. When shears are computed, a "taper effect" must be considered.

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On Maximum Gain-Band Width Product in Amplifiers

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By introducing an electrostatic problem analogous to the impedance function problem it is found possible to prove that, under simple and quite broad assumptions, the maximum possible gain-band width products for two-terminal coupling, four-terminal low-pass coupling and four-terminal band-pass coupling are, respectively, 1, 2.47, and 2.53 times $g/\pi C$. The first result was conjectured by Wheeler long ago, and it and the second have recently been proved by Bode. The last result covers a case more general than those considered by Bode.

INTRODUCTION

A GENERALIZED circuit diagram of a single stage of a commonly used type of triode amplifier which we wish to consider here is shown in Fig. 1. The "coupling impedance" is made up of the capacity $C=C_p+C_g$ associated with the tube, in parallel with whatever impedance one chooses to add, as represented by the two-terminal box in the diagram. Thus, and this is the fundamental assumption, the gain must vary like $1/\omega$ at large ω. Coupling of this type is known as "two-terminal." It is also possible to separate the grid and plate capacities, the connection then being made by a box with four terminals, and with a corresponding change in description.

Sometimes the aim is to make the gain of the amplifier large from zero up to some maximum frequency; in other cases one aims at an amplification which is large in the vicinity of some center frequency. The former is called a low pass, the latter a band-pass amplifier. There is a simple connection between the two types, at least in the case where the pass-band width is small compared to the mean frequency, for then one can construct a band-pass amplifier from a given low pass one by simply replacing each condenser by a shunt resonant circuit and each inductance by a series resonant circuit. On the other hand, not all band-pass structures have realizable low pass equivalents.

Now almost always one wants a high gain per stage G and in many applications a large band of frequencies $\Delta\nu$ is also a necessity. Plainly these requirements are contradictory for a large gain implies a high impedance, whereas the condenser formed by the tube capacities has an impedance which decreases with increasing frequency. We may put this idea in quantitative form as follows. Consider any given coupling network and let all the resistances be increased by a factor n, the inductances by n^2, and let the condensers be left unchanged. Then the impedance at (radian) frequency ω will plainly be exactly n times what it was before at frequency $n\omega$. Thus such a change increases the gain and decreases the band width by exactly the same factor, the product remaining constant.

The value of this constant depends, of course, on the tube characteristics, the type of coupling

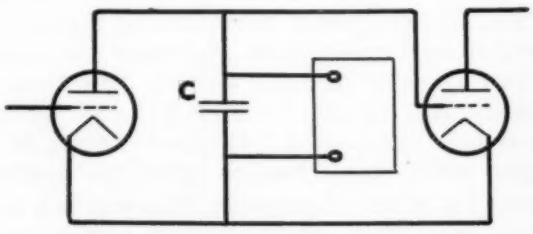

FIG. 1. This figure shows a typical single stage of an amplifier as considered in the text. The coupling impedance consists of the condenser C in parallel with a two terminal impedance which is arbitrary, except that its impedance does not approach zero at $\omega\to\infty$. Otherwise said, *all* the shunt capacity is contained in C. The minimum possible value of C is C_p+C_g.

impedance, and the precise definition of "band width."

As to the term band width, we define it to mean the frequency range over which the gain is *constant*, this being measured from one side of the pass band to the other in the band-pass case, and from zero up in the low pass case. We choose this rather narrow definition because without it, simple results are not possible. For if the gain is not constant over some region, a many-stage amplifier will have a different band width than a single and $\Delta\nu$ becomes a function of n. Obviously this point is of practical importance too, for loss of band width with increasing n is a serious matter. On the other hand, our definition may be too narrow to be justified on any practical grounds. In fact, there are often reasons for wanting the amplifier as a whole to have a pass characteristic with rounded corners, although this still usually means that the response for any given stage will be nearly square. In any case, once the desired shape is specified the methods to be described can be used to determine how best to get it. The real justification must be as above —with it simple results are possible, without it not.

A further point which may be mentioned here as well as anywhere else relates to what we mean by "constant." As might be guessed, no finite number of circuit elements can give a strictly constant gain. But the circuits we consider will approach this as the number of circuit elements becomes large. In the interests of simplicity we have omitted consideration of the limiting process. Also, as will be observed presently, we have taken the poles and zeros of the impedance function to lie on the real axis, which is where they would be in the limit, but not where they should be in practice, with a finite number of circuit elements. Another paper will discuss this limiting process and other practical matters in some detail.

As to the tube characteristics, it is easy to see that the mutual conductance g and the total capacity C are the things of importance in the two-terminal case, and that they enter through the ratio g/C, which is a measure of the merit of the tube for purposes of broad band amplification.

Thus the gain-band width product is some constant times g/C, with the constant depending on the coupling network.

The question that immediately arises, and with which this paper is concerned, is—what is the largest possible value of this numerical coefficient and how is it achieved?

It was shown by Wheeler that in the two-terminal low pass case one could achieve $|G|\Delta\nu = g/\pi C$ while in the four-terminal case one could achieve $2g/\pi C$ if the grid and plate capacities are equal. The results in the band-pass case are similar, except that the latter result is multiplied by

$$(C_p+C_g)/2[C_pC_g]^{\frac{1}{2}}$$

which is unity when $C_p=C_g$ and slightly greater for moderate deviations from equality.

These important results were obtained by Wheeler by showing that certain specific and cleverly chosen coupling impedances gave the above-quoted numbers. But no proof was given that these numbers are the maximum possible, though Wheeler conjectured that this was so.

In the present paper we wish to show six things. First, the highest gain-band width product attainable by any two-terminal coupling circuit is $g/\pi C$, as given by Wheeler. Second, Wheeler's four-terminal result is also the best possible for a single stage. Third and fourth, that the same result, for the two-terminal case, may be obtained in a much simpler way by making the various stages in the amplifier non-identical, but that no better can be done. And finally, for the four-terminal band-pass case, non-identical stages are simpler and give $2.53g/\pi C$ and no better than this can be done.

Broadly speaking, the method of attack is to reduce the circuit theory problem to one of electrostatics by means of a simple transformation, whereupon the problem is solved by the use of potential theory.

The easiest case to understand is that of two-terminal coupling with non-identical single-tuned stages and this is, therefore, explained first. There follow discussions of the two-terminal case with arbitrary coupling, the arbitrary four-terminal case, and finally the four-terminal band-pass case with non-identical stages.

STAGGER-TUNED TWO-TERMINAL COUPLING

This phrase "stagger-tuned two-terminal coupling" is intended to describe a band-pass amplifier any two stages of which are coupled by the simplest possible network, as shown in Fig. 2, but in which the various stages in general have different values of L and R, i.e., are tuned to different (complex) frequencies.

The gain of some specific stage is easily found to be

$$G=-\frac{jg}{C}\frac{\omega}{(\omega-\omega_p)(\omega+\bar{\omega}_p)}, \qquad (1)$$

where ω_p is the complex resonant frequency of the pth circuit where p goes from 1 to n. Now if the center of the pass band is at a frequency large compared to the deviations therefrom, as we will hereafter assume, one can neglect the variations owing to the zero at the origin and the pole at $-\omega_p$. If we do this, and set up a new system of coordinates with origin at the band center, we find the gain of the whole amplifier to be

$$G^n \cong \left(-\frac{jg}{2C}\right)^n \frac{1}{\Pi(\omega-\omega_p)}, \qquad (2)$$

where G is the nth root of the gain of the n-stage amplifier and so is a sort of mean stage gain, and ω_p is now the resonant frequency of a given stage, *measured from band center*.

Now, if we consider that the various ω_p can be given any complex values whatever by suitably choosing R and L, our problem becomes that of finding the distribution which gives the highest constant value of $|G|$ on a chosen section of the real ω axis. Of course, negative values of R are not practical, but formally this makes no difference, for an ω_p with a negative imaginary part will be found to give the same contribution to the gain as one with a positive imaginary part.

To answer this question, we take the logarithm of Eq. (2), the real part of the resulting equation being, after division by n,

$$\log |G| = \log \frac{g}{2C} - \frac{1}{n} \sum \log|\omega-\omega_p|. \qquad (3)$$

We see that the second term on the right is of the same form as the potential resulting from n line charges at the points ω_p, each carrying a charge per unit length equal to const./n.

From this point of view, the answer to the question of how to choose the ω_p so that $|G|$ shall be constant within the pass band becomes almost obvious. For consider a conducting cylinder of cross section which is arbitrary except that it has mirror symmetry about a horizontal line. If now we charge this cylinder, the potential inside will be constant. If next we remove the half of the cylinder that lies below the plane of symmetry, but do not allow the charge to redistribute on the upper half, the potential will still be constant on that part of the symmetry plane that is "inside" the remaining half-cylinder. If we now draw, in the ω plane, a reproduction of the cylinder cross section, using the real axis as the symmetry line, we have only to put the poles ω_p of the gain function along the upper half of the map of the cylinder and with density corresponding to the density assumed by the charge distribution, whereupon the gain along that part of the real axis spanned by the pole distribution will be constant.

Plainly, since all the poles are above the real axis, the distribution is physically realizable. We have therefore established an infinite class of amplifiers, all of which give constant gain throughout the pass band. Furthermore, this class includes all possible such amplifiers, for if we attempt to use a charge distribution which has a gap in it, we see that in the electrostatic analog, some lines of force will come through the gap, so that the potential cannot be constant inside.

We have therefore found all possible ways of making two-terminal stagger-tuned amplifiers with flat pass bands and it remains only to discover which of these gives the greatest possible gain-band width product.

The answer to this question is obviously to say that the closer the charges are brought to the axis, the higher the potential they will make there so that the cylinder should become a flat ribbon just spanning the pass band. The charge distribution on such a flat ribbon is well known, or can be simply found by a solution of Laplace's equation in elliptic cylinder coordinates. The result is that if ω_1 is the full width of the pass band, the charge density varies like

$$\{1-(4\omega^2/\omega_1^2)\}^{-\frac{1}{2}}$$

in order to give constant potential in $|\omega| < \omega_1/2$.

Returning now to Eq. (3) we have only to find the mean value of $\log(\omega-\omega_p)$, using the above density function.

$$\log |G| = \log \frac{g}{2C} - \frac{\int_0^{\omega_1/2} \log \omega \frac{d\omega}{[1-(4\omega^2/\omega_1^2)]^{\frac{1}{2}}}}{\int_0^{\omega_1/2} \frac{d\omega}{[1-(4\omega^2/\omega_1^2)]^{\frac{1}{2}}}}$$

$$= \log \frac{g}{2C} + \log \frac{4}{\omega_1}, \qquad (4)$$

and so

$$|G|\Delta\nu = g/\pi C \qquad (5)$$

is the maximum possible gain-band width obtainable by means of staggered single-tuned coupling.

The above is for the band-pass case. The same result may be had in the low pass case by alter-

nating stages using *RC* coupling with stages using "shunt-peaked" coupling.

The question now naturally arises as to whether one cannot do better yet in the two-terminal case by admitting more complex networks in each stage, while still keeping to the general idea of having the various stages differently coupled.

The answer is no; and while a full proof will not be given here, we can usefully mention the general idea of a proof, leaving it to the reader to fill in the details after reading the next section.

If we introduce more reactances into the coupling impedance, the general result is to add more poles and zeros to the impedance function but it is easy to see that the total number of poles in the pass band must exceed the number of zeros by exactly one. Then in the potential theory analog, where the poles correspond to positive charge and the zeros to negative, we always have the same total amount of charge and so can expect to get the same potential and so the same gain. This might not be so if it were possible to get the poles to lie nearer to the real axis than the zeros as this would make more lines of force leave the bottom of the semicylinder than the top and so raise the potential on the real axis. But, as will be shown in detail in the next section, such a pole and zero distribution is impossible.

The conclusions are, then, that simple stagger-tuned two-terminal coupling can reach the same value of gain-band width product as that for the two-terminal filter coupling of Wheeler and that no better stagger-tuned scheme is possible.

ARBITRARY TWO-TERMINAL COUPLING

We next answer the question—what is the best that can be done with a single stage and an arbitrary two-terminal coupling network?

For the sake of variety we will work this out for the low pass case. As remarked above, the number of poles must exceed the number of zeros by one in order to give the correct behavior at infinity and so the gain is of the form

$$G=-\frac{jg}{C}\frac{\Pi(\omega-\omega_z)}{\Pi(\omega-\omega_p)}, \qquad (6)$$

where the number of poles ω_p exceeds the number of zeros ω_z by one. As before, we take the logarithm and so find

$$\log|G|=\log(g/C)+\sum\log|\omega-\omega_z| -\sum\log|\omega-\omega_p|. \qquad (7)$$

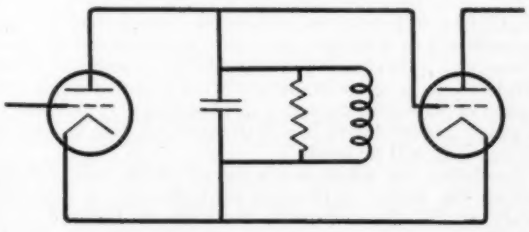

FIG. 2. Representative stage of a "stagger-tuned" amplifier. The inductance and resistance values L and R are, in general, different for the various stages.

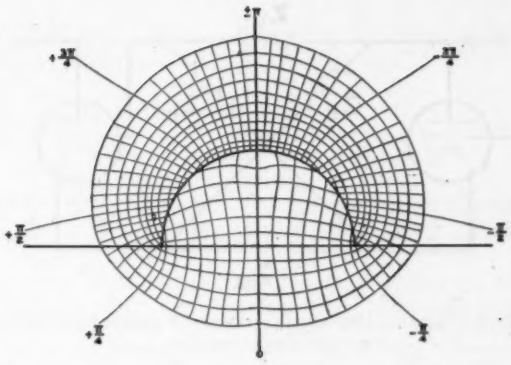

FIG. 3. Typical lines of constant log $|G|$ and constant phase in the ω plane. The particular set of lines shown was drawn for the case in which the poles are uniformly distributed on a semicircle above the real axis, and the drawing is moderately accurate.

Now the only new question is—what prevents us from putting the poles on one cylinder and the zeros on another which extends further above the real axis? If it were possible, this would obviously raise the gain-band width product. In the electrostatic analog this would correspond to a charge plus a double layer on the cylinder, with the sign of the double layer chosen so as to make more lines of force leave the lower surface of the semicylinder than the upper, so raising the potential on the lower side.

That this cannot happen is shown as follows. As we have seen, the lines of constant potential in the electrostatic problem correspond to lines of constant log $|G|$ in the ω plane. Obviously enough, the orthogonal lines in the two cases are, respectively, the field lines and the lines of constant phase. Such lines are illustrated qualitatively in Fig. 3, which shows lines diverging from a region above the real axis which bounds all the poles and zeros, though the exact location within this boundary is not specified. This figure also shows lines of constant log $|G|$. Now we know that at $\omega\to\infty$ the lines of equal phase ϕ

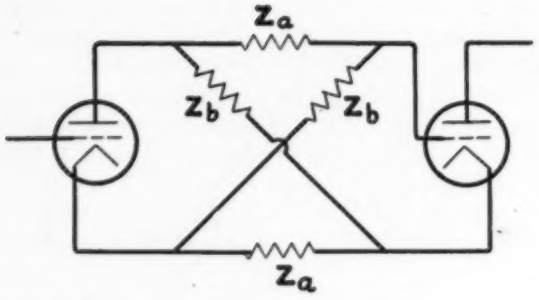

FIG. 4. Diagram illustrating the most general possible four-terminal coupling.

must run radially and be evenly spaced in angle. Consider then the line $\phi=-\pi/2$ in its course inward from infinity where it is asymptotic to the real axis. This line *must* come down on the top of the region bounding the charge as illustrated in Fig. 3. For if it did not, an adjacent line $\phi=-[(\pi/2)+\epsilon]$ would necessarily cross the real axis, which is impossible since it would mean negative resistance in shunt with C. Thus the fraction of the lines leaving the bottom of the region is less than or equal to one-half, the equality being attained when all the charges are on the axis, when $\phi=\pm\pi/2$ everywhere on the real axis, except in the pass band. Plainly the more lines leave on the bottom, the higher the potential at $\omega=0$ and, since we have already found that a charge distribution with all charge on the axis and giving constant potential is possible we see that the highest possible gain-band width product will be attained when all poles and zeros are on the real ω axis. Also, this product will have the value computed in the preceding section.

The only further point is that, although we have found the *best* pole and zero distribution, and have shown that it is not obviously impossible, neither have we shown that it is physically realizable. This is done simply enough by observing that the special case of the use of an infinite ladder filter for a coupling impedance, as discovered by Wheeler does actually give the desired distribution (and also the computed gain-band width product). A similar proof holds in the band-pass case.

The conclusion then is that no two-terminal coupling network can do better than $|G|\Delta\nu=g/\pi C$.

ARBITRARY FOUR-TERMINAL COUPLING

Consider next arbitrary four-terminal coupling, and let us again consider the low pass case.

As is well known, the most general physically realizable four-terminal impedance may be written in lattice form, as is Fig. 4, where using a symmetrical lattice amounts to assuming* that $C_g=C_p=C/2$ or $(1/2)C$. As a result we have, at $\omega\rightarrow\infty$,

$$Z_a+Z_b\leqq-4j/C\omega, \quad \omega\rightarrow\infty. \quad (8)$$

Also, the transfer impedance, which controls the gain, is

$$Z_{12}=Z_a-Z_b. \quad (9)$$

Now it is easy to see that the highest gain-band width product will be obtained if Z_a and Z_b have the same value at infinity, i.e., $-2j/C\omega$, and this will be assumed from here on. Also, we will save time by assuming from the start that the best place for the poles and zeros is on the real axis—the proof is like that in the preceding section.

Now if we regard the magnitudes and phases of Z_a and Z_b as independent variables we see that, for fixed magnitudes, the best phases would give Z_a and Z_b phases differing by π, which for realizable circuits, can only mean $\phi_a=\pm\pi/2$, $\phi_b=\mp\pi/2$. This can be done in two ways: (a) by making the zeros of Z_a coincide with the poles of the Z_b and *vice versa* and (b) by confining the poles and zeros of Z_a to a series of pass bands separated by stop bands, while Z_b has pass bands corresponding to the stop bands of Z_a and *vice versa*.[1] Considering first possibility (a) this can be done in the manner illustrated qualitatively in Fig. 5 where the poles and zeros, represented by crosses and circles are indicated on an ω scale. Also shown are the Foster decompositions for Z_a and Z_b. This figure shows no poles and zeros outside the pass band, and this will also be true of our proof, but no essential changes are needed to include this possibility, nor is the end result changed.

Now Z_a-Z_b has poles wherever either Z_a or Z_b have them and a little thought will show that

* If $C_g\neq C_p$ then we can gain a further factor $(C_g+C_p)/2\times(C_gC_p)^{\frac{1}{2}}$ by following the symmetrical lattice by an ideal transformer. For the band-pass case this ordinarily presents no practical difficulties. For the low pass case, it is impossible to go to zero frequency with a transformer, but one can, in principle, approach as closed as may be desired. The point involved here is well understood, and no further mention of it will be made.

[1] Possibility (*b*), which is the better of the two, did not occur to the writer until after he had seen, in the Vail Library at the Massachusetts Institute of Technology, a copy of a set of notes by Dr. H. W. Bode. Chapter XVII of these notes attacks the present problem, though in a completely different manner. It is understood that these notes will soon be published in book form.

there are no zeros in between. Therefore $Z_a - Z_b$ has the form

$$Z_a - Z_b = \text{constant}/\Pi(\omega - \omega_p), \qquad (10)$$

as is also indicated in the lower line of Fig. 5.

Now we already know how best to arrange the ω_p, so if we knew the constant, the problem would be solved.

This is done as follows. We can write

$$Z_a = -\frac{2j}{C}\frac{\Pi(\omega-\omega_o)}{\Pi(\omega-\omega_e)}, \qquad (11)$$

$$Z_b = -\frac{2j}{C}\frac{\Pi'(\omega-\omega_e)}{\Pi(\omega-\omega_o)}, \qquad (12)$$

where the subscripts o and e mean even and odd, respectively, and refer to the numbers of the poles in Z_{12}, as shown in the last line of Fig. 5. Also, the prime on the product sign in Eq. (12) indicates that the outermost factors are omitted.

Now we simply subtract (12) from (11) whereupon we get a fraction with what is apparently a complicated numerator. But from (10) the numerator must actually be a constant, so that higher powers of ω vanish. Then the constant term is easily found and so

$$Z_{12} = -\frac{2j}{C}\frac{[\Pi\omega_o]^2}{\Pi(\omega-\omega_o)\Pi(\omega-\omega_e)}. \qquad (13)$$

Now we arrange the poles so that their density varies like $(1-\omega^2/\omega_1^2)^{\frac{1}{2}}$ inside ω_1, knowing that this will give constant Z_{12} there, and evaluate Z_{12}

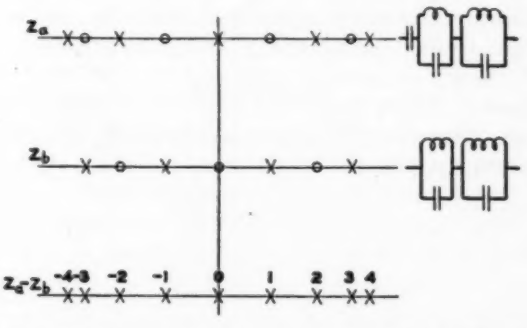

FIG. 5. Diagrammatic representation of the location of poles and zeros for the amplifier of Fig. 4. The horizontal lines are ω scales and circles and crosses denote zero and pole positions for the various impedances indicated on the left-hand ends of the ω scales. Also indicated are Foster decompositions capable of attaining these pole and zero arrangements. In the last line, the numbers illustrate the notation used in identifying the poles as in Eqs. (11) and (12).

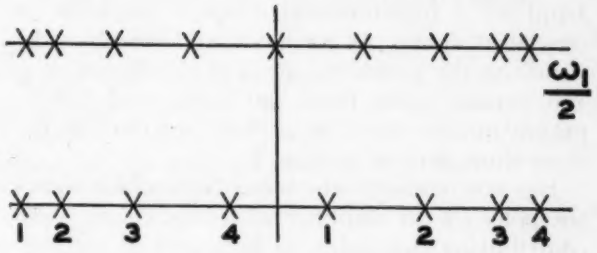

FIG. 6. Pole distribution diagrams, similar to Fig. 5. The upper line is for a single-stage band-pass amplifier having nine poles within the pass band. The origin corresponds to band center and the pass band extends from $-\omega_1/2$ to $+\omega_1/2$. The lower diagram shows a distribution of eight poles as obtained by using four stages, each contributing two poles. The poles corresponding to the various stages are indicated by the numbers.

at $\omega = 0$, noting that there is one more factor in the denominator than the numerator.

This is done as before, the result being

$$|G| = 4/C\omega_1, \qquad (14)$$

and so $G\Delta\nu = 2g/\pi C$ is the best that can be done with scheme (a). Obviously this limit can be reached, and it is also easy to show that the same result holds in the band-pass case.

If now we consider the arrangement of poles and zeros suggested in scheme (b) we find that the above method is not very convenient. But the problem may be handled by using the fact that, for maximum constant gain, the transfer impedance must be of the form

$$\text{const} \times \exp\{n[(1-\omega^2)^{\frac{1}{2}} - i\omega]\},$$

getting alternate loops of the real part of this from Z_a and Z_b, respectively, and evaluating the impedance level from the easily proved theorem that the integral of the real part of Z_a must be equal π/C.

This has been worked out in detail by Bode, with the result that $G\Delta\nu = (\pi^2/4)(g/\pi C)$. This exceeds the result of case (a) by $\pi^2/8$ and is apparently the best that can be done for a single stage.

FOUR-TERMINAL STAGGER-TUNED COUPLING

At first sight, it might appear that the above represents the absolute limit, but close consideration will show that notably better can be done, in the band-pass case, by the use of stagger tuning with simple coupling circuits.

Thus consider Fig. 6 which shows on the upper line the poles of Z_{12} in the vicinity of the pass

band of a four-terminal coupled amplifier as described above. As we have seen, the gain depends on the geometric mean of the distances of the various poles from the origin and this is plainly notably less than $\omega_1/2$; in fact our calculations show it to be just $\omega_1/4$.

But now consider the lower line, which shows the poles for an amplifier with four stages, each contributing two poles, as indicated in the diagram. Again the gain depends on the constant which now depends on the geometric mean of the distances 1–1, 2–2, etc. It may be shown by simply writing down the analysis that all the other constants are the same, so we have only to ask about this mean value. But, this mean is greater than $\omega_1/4$, since every factor exceeds that value. So this type of amplifier can achieve a higher gain-band width product than that treated in the preceding section.

To evaluate this mean, it is convenient to introduce a parameter θ, such that $\omega=(\omega_1/2)\sin\theta$, so that a distribution of poles that is uniform in θ will have density like

$$(1-4\omega^2/\omega_1^2)^{-\frac{1}{2}}$$

in ω. Then in the earlier case we were interested in the mean value of $\log 2\sin\theta$ whereas now we want the mean value of $\log(\sin\theta+\cos\theta)$. The first integral is well known and has already been recorded above. The other can be evaluated by series and when this is done the exponential of the difference of the two logarithmic means, which is the ratio of the gain-band width products, comes out at 1.26687.

Thus, in this way, one can achieve $|G|\Delta\nu=2.53(g/\pi C)$ which is noticeably more than Wheeler's "maximum."

This cannot be done in the low pass case, since there the poles for any stage must be symmetrical with respect to the origin.

Also, more complicated circuits used in the stagger tuned fashion are worse. For example, if we try to get the nine poles of the upper part of Fig. 6 by three stages, each contributing three poles, the best arrangement distributes the poles among the stages like this:

$$(-4,\ -1,\ +2)(-3,\ 0,\ +3)(-2,\ +1,\ +4),$$

and this decreases the mean pole spacing of each stage by bringing it closer to the situation of the preceding section.

We conclude that, for the band-pass case, $|G|\Delta\nu=2.53(g/\pi C)$ can be done, but not better.

CONCLUSION

In this paper we have established, by methods which are simple, and appear to be rigorous, the maximum possible gain-band width products. These are $G\Delta\nu=g/\pi C$ for two-terminal coupling, $2.53g/\pi C$ for four-terminal band pass and $2g/\pi C$ for four-terminal low pass.

The fundamental assumption on which this is all based is that the gain at large frequencies goes down like $1/C\omega$ for a single stage, or more generally like $(1/C\omega)^n$ for an n-stage amplifier. A corresponding assumption is made for the four-terminal case. While this is certainly the case in all ordinary amplifiers, cases are conceivable in which it may not be so—in such cases the present theory does not apply.

In this paper we have completely disregarded all practical questions and have simply asked—what is the *limit*, regardless of anything. For example, one important question is, where does one put the poles when only a finite number are available, and what happens to the gain as a result of the change in distribution? Other important questions relate to simplicity, ease and stability of tuning, etc. Also, one may not want a flat topped pass band. Plainly the present type of theory is easily extended to any assigned desired shape, but what are the results? These and other practical questions will be treated in another paper.

A "Power Function" for the Determination of Lagrangian Generalized Forces

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The purpose of this paper is to present the development of a "power function," P, such that Lagrangian generalized forces, F_{q_r}, which stem from a wide variety of forces (as, externally applied, conservative, dissipative forces proportional to the nth power of speed, etc.) can be obtained from the simple relation:

$$F_{q_r}=\partial P/\partial \dot{q}_r.$$

The value of this easily obtained function lies in the fact that it places the task of determining almost all generalized forces which occur in practice on the same basis as that of determining conservative forces from a potential-energy function. Moreover, it is equally applicable to mechanical, electrical, and electro-mechanical systems. Since it is believed that the function has wide applications, several of its most useful forms are listed. Two specific examples are given.

INTRODUCTION

LAGRANGIAN generalized forces[1] for any holonomic system are obtained by evaluating the right-hand side of

$$F_{q_r}=\sum_{i=1}^{p} F_{x_i}\frac{\partial x_i}{\partial q_r}+F_{y_i}\frac{\partial y_i}{\partial q_r}+F_{z_i}\frac{\partial z_i}{\partial q_r}, \qquad (1)$$

where q_r is any one of the n independently variable generalized coordinates of the system and F_{x_i}, F_{y_i}, F_{z_i} are the rectangular components of force (including all types except those of constraint) acting on the ith particle of the system of which there are p in number. This task is straightforward but may become quite involved if the individual forces are of such a nature that they cannot be taken care of with a potential energy function.

A study of this problem has revealed the fact that, for almost all types of forces encountered in practice, it is possible to set up a "power function," P, from which generalized forces can be obtained by the simple relation,

$$F_{q_r}=\partial P/\partial \dot{q}_r.$$

As will presently be shown, P is usually quite easy to obtain (standard forms can be written for the most frequently encountered types of forces) and the general method is applicable to electrical, mechanical, and electro-mechanical systems.

DEVELOPMENT OF THE POWER FUNCTION

Consider the quantity φ defined by

$$\varphi=\sum_{i=1}^{p} F_{x_i}d\dot{x}_i+F_{y_i}d\dot{y}_i+F_{z_i}d\dot{z}_i, \qquad (2)$$

where $\dot{x}_i$, $\dot{y}_i$, $\dot{z}_i$ represent the rectangular components of velocity of the ith particle of the system and F_{x_i}, F_{y_i}, F_{z_i} have the same meaning as in (1).

From the transformation equations relating rectangular and generalized coordinates,

$$x_i=f_i(q_1, q_2\cdots q_n, t);\quad y_i=g_i(q_1, q_2\cdots q_n t);$$
$$z_i=h_i(q_1, q_2\cdots q_n, t),$$

we obtain at once,

$$\dot{x}_i=\frac{\partial x_i}{\partial q_1}\dot{q}_1+\frac{\partial x_i}{\partial q_2}\dot{q}_2+\cdots\frac{\partial x_i}{\partial q_n}\dot{q}_n+\frac{\partial x_i}{\partial t} \text{ etc. for } \dot{y} \text{ and } \dot{z}. \qquad (3)$$

Now, *regarding all coordinates and time as constant* we get from (3),

$$d\dot{x}_i=\frac{\partial x_i}{\partial q_1}d\dot{q}_1+\frac{\partial x_i}{\partial q_2}d\dot{q}_2+\cdots\frac{\partial x_i}{\partial q_n}d\dot{q}_n,$$

and similar expressions for $d\dot{y}_i$ and $d\dot{z}_i$. Substituting these quantities into (2) and collecting terms, the result is

$$\varphi=\sum_{i=1}^{p}\left(F_{x_i}\frac{\partial x_i}{\partial q_1}+F_{y_i}\frac{\partial y_i}{\partial q_1}+F_{z_i}\frac{\partial z_i}{\partial q_1}\right)d\dot{q}_1$$
$$+\sum_{i=1}^{p}\left(F_{x_i}\frac{\partial x_i}{\partial q_2}+F_{y_i}\frac{\partial y_i}{\partial q_2}+F_{z_i}\frac{\partial z_i}{\partial q_2}\right)d\dot{q}_2$$
$$+\ \cdots\cdots\cdots\cdots$$
$$+\sum_{i=1}^{p}\left(F_{x_i}\frac{\partial x_i}{\partial q_n}+F_{y_i}\frac{\partial y_i}{\partial q_n}+F_{z_i}\frac{\partial z_i}{\partial q_n}\right)d\dot{q}_n. \qquad (4)$$

Let us now assume for the moment that there exists a function, P, such that

$$F_{x_i}=\frac{\partial P}{\partial \dot{x}_i},\quad F_{y_i}=\frac{\partial P}{\partial \dot{y}_i},\quad F_{z_i}=\frac{\partial P}{\partial \dot{z}_i}. \qquad (5)$$

[1] E. T. Whittaker, *Analytical Dynamics* (Cambridge University Press, New York, 1927), pp. 34–36.

Substituting these relations, together with the fact easily seen from (3) that $\partial x_i/\partial q_r=\partial \dot{x}_i/\partial \dot{q}_r$, etc., into (4) gives

$$\varphi=\sum_{i=1}^{p}\left(\frac{\partial P}{\partial \dot{x}_i}\frac{\partial \dot{x}_i}{\partial \dot{q}_1}+\frac{\partial P}{\partial \dot{y}_i}\frac{\partial \dot{y}_i}{\partial \dot{q}_1}+\frac{\partial P}{\partial \dot{z}_i}\frac{\partial \dot{z}_i}{\partial \dot{q}_1}\right)d\dot{q}_1$$

$$+\ \cdots\cdots\cdots\cdots$$

$$+\sum_{i=1}^{p}\left(\frac{\partial P}{\partial \dot{x}_i}\frac{\partial \dot{x}_i}{\partial \dot{q}_n}+\frac{\partial P}{\partial \dot{y}_i}\frac{\partial \dot{y}_i}{\partial \dot{q}_n}+\frac{\partial P}{\partial \dot{z}_i}\frac{\partial \dot{z}_i}{\partial \dot{q}_n}\right)d\dot{q}_n. \quad (6)$$

But this is just

$$\frac{\partial P}{\partial \dot{q}_1}d\dot{q}_1+\frac{\partial P}{\partial \dot{q}_2}d\dot{q}_2+\cdots\frac{\partial P}{\partial \dot{q}_n}d\dot{q}_n=dP. \quad (7)$$

Comparing (1) and (4) we see that the coefficients of $d\dot{q}_1$, $d\dot{q}_2\cdots d\dot{q}_n$ are F_{q_1}, $F_{q_2}\cdots F_{q_n}$, respectively. From a comparison of (4) and (6) it follows that; $F_{q_1}=\partial P/\partial \dot{q}_1$, $F_{q_2}=\partial P/\partial \dot{q}_2$, etc., or in general

$$F_{q_r}=\frac{\partial P}{\partial \dot{q}_r}. \quad (8)$$

Furthermore a comparison of (7) and (2) shows that

$$dP=\sum F_x d\dot{x}+F_y d\dot{y}+F_z d\dot{z}. \quad (9)$$

Returning to the assumption contained in (5) we see that by a simple process of differentiation we can write

$$\frac{\partial F_{x_i}}{\partial \dot{y}_j}=\frac{\partial^2 P}{\partial \dot{y}_j\partial \dot{x}_i};\quad \frac{\partial F_{y_j}}{\partial \dot{x}_i}=\frac{\partial^2 P}{\partial \dot{x}_i\partial \dot{y}_j}, \text{ etc.}$$

Thus it follows that if (5) is true the following relations must hold,

$$\frac{\partial F_{x_i}}{\partial \dot{y}_j}=\frac{\partial F_{y_j}}{\partial \dot{x}_i};\quad \frac{\partial F_{x_i}}{\partial \dot{z}_j}=\frac{\partial F_{z_j}}{\partial \dot{x}_i};\quad \frac{\partial F_{y_i}}{\partial \dot{z}_j}=\frac{\partial F_{z_j}}{\partial \dot{y}_i}. \quad (10)$$

These relations constitute necessary and sufficient conditions that the right-hand side of (9) be exact. Hence in any particular problem, for which relations (10) are satisfied, P is given by

$$P=\int\sum_{i=1}^{p}F_{x_i}d\dot{x}_i+F_{y_i}d\dot{y}_i+F_{z_i}d\dot{z}_i, \quad (11)$$

and, having expressed the result of this integration in generalized coordinates, the generalized forces are obtained at once from (8). In performing this integration it must be remembered that, in accord with the method of development, *all coordinates and time are to be regarded as constants.*

It may be that in looking over a particular problem we see that the active forces can be expressed very conveniently in terms of certain coordinates which are not necessarily rectangular and which perhaps are not the ones we would most like to appear in the final differential equations of motion of the system. If that is the case, we can write an integral expression for P which may be, and generally is, considerably more convenient than (11). Let us refer to these coordinates as g_1, $g_2\cdots g_s$ (which may be greater in number than, n, the degrees of freedom) and the corresponding forces as f_1, $f_2\cdots f_s$, which must include all forces acting on all parts of the system except forces of constraint. By a development exactly similar to that which lead to (7) we can write

$$P=\int f_1 d\dot{g}_1+f_2 d\dot{g}_2+\cdots f_s d\dot{g}_s. \quad (12)$$

Now to put this in usable form containing just the correct number of desired coordinates q_1, $q_2\cdots q_n$, we merely eliminate the g's and $\dot{g}$'s by means of transformation equations and equations of constraint. It must be remembered that f_r, any one of the forces in (12), is determined by $\delta W_{g_r}=f_r\delta g_r$ where δW_{g_r} is the increment of work done by f_r when g_r *alone* is changed to the extent of $+\delta g_r$.

CERTAIN ESPECIALLY USEFUL FORMS OF P

Let us first consider the rather broad class of dissipative forces which are of such a nature that the force on an individual particle m_i moving with speed $\dot{S}_i$ is opposite in direction to its motion and given in magnitude by $a_i\dot{S}_i^n$, where n is any power including zero and a_i may be constant or a function of coordinates and/or time.

It is clear that the x component of this force is given by,

$$f_{x_i}=-a_i\dot{S}_i^n\left(\frac{\dot{x}_i}{\dot{S}_i}\right)=-a_i\dot{S}_i^{n-1}\dot{x}_i$$

and similar expressions for f_{y_i} and f_{z_i}. Since $\dot{S}_i^2=\dot{x}_i^2+\dot{y}_i^2+\dot{z}_i^2$, it is seen that relations (10) are satisfied for all values of n. Thus, applying (11),

$$P=-\int\sum_{i=1}^{p}a_i\dot{S}_i^{n-1}(\dot{x}_i d\dot{x}_i+\dot{y}_i d\dot{y}_i+\dot{z}_i d\dot{z}_i), \quad (13)$$

which easily integrates to

$$P=-\sum_{i=1}^{p}\frac{a_i}{n+1}\dot{S}_i^{n+1}. \tag{14}$$

This form of P is easily expressed in terms of any desired generalized coordinates. An application of (8) to (14) immediately gives generalized forces which stem from friction, viscous drag, etc., depending on the value of n. For ordinary dynamic friction $n=0$. For viscous drag $n=1$. (Note that for $n=1$, (14) reduces to the well known Rayleigh "dissipation function.")[2] For forces of the type experienced by objects moving rapidly through a fluid n is greater than one.

When it is assumed that $n=-1$ (13) becomes,

$$P=-\sum_{i=1}^{p}a_i\log\dot{S}_i. \tag{15}$$

Another very useful form,

$$P=\sum_{i=1}^{p}F_{xi}\dot{x}_i+F_{yi}\dot{y}_i+F_{zi}\dot{z}_i, \tag{16}$$

is obtained from (11) by assuming that all forces are independent of velocity. It can be applied directly to conservative as well as various types of externally applied forces.

The following form obtained from (12),

$$P=f_1\dot{q}_1+f_2\dot{q}_2+\cdots f_s\dot{q}_s, \tag{17}$$

is equivalent to (16) and generally more convenient. It should be borne in mind that F_{xi}, F_{yi}, F_{zi} and $f_1, f_2\cdots f_s$ may be functions of coordinates and/or time.

When there is a dissipative force on any one particle m_i proportional to the nth power of its speed v_{il} relative to any other particle m_l, it follows without difficulty that,

$$P=-\sum\frac{a_{il}}{n+1}v_{il}^{n+1}. \tag{18}$$

In this expression v_{il} is always to be regarded as a positive quantity and the sum includes each relative speed one time only. For p particles there are $p(p-1)/2$ such relative speeds.

Other useful forms of P are easily obtained. It is clear, of course, that for a system acted upon by two or more types of forces, the proper power function is merely the sum of power functions corresponding to the individual types.

[2] See reference 1, page 230.

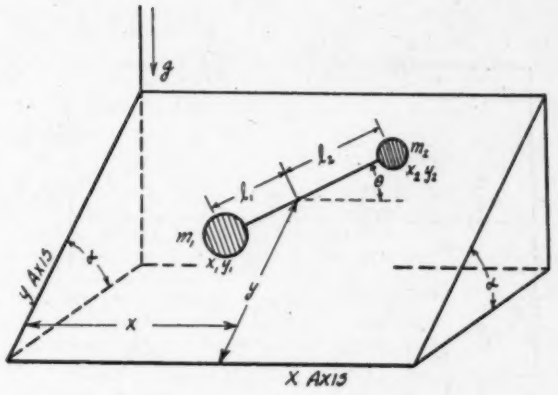

FIG. 1. Mechanical system on which dissipative and conservative forces are acting.

EXAMPLES ILLUSTRATING THE USE OF P

(1) Consider two small masses m_1 and m_2 connected with a light rigid rod and moving in contact with an inclined plane as shown in Fig. 1. This is a problem of three degrees of freedom and we shall find, by means of a P function, the generalized forces corresponding to coordinates x, y, and θ where x, y might, if so desired, locate the center of gravity of the dumb-bell. Assume gravity acting in the direction indicated and further, because of the contact of the masses with the plane, assume a drag on m_1 given in magnitude by $a\dot{S}_1^{n_1}$ and a drag on m_2 given by $b\dot{S}_2^{n_2}$. It follows from (14) and (16) that;

$$P=-\frac{a(\dot{x}_1^2+\dot{y}_1^2)^{(n_1+1)/2}}{n_1+1}-\frac{b(\dot{x}_2^2+\dot{y}_2^2)^{(n_2+1)/2}}{n_2+1}$$
$$-m_1g\sin\alpha\dot{y}_1-m_2g\sin\alpha\dot{y}_2,$$

where x_1, y_1 and x_2, y_2 are the rectangular coordinates of m_1 and m_2, respectively. But from the figure it is seen that, $x_1=x-l_1\cos\theta$; $y_1=y-l_1\sin\theta$; $x_2=x+l_2\cos\theta$; $y_2=y+l_2\sin\theta$. Thus P can be written as,

$$P=-\frac{a}{n_1+1}[(\dot{x}+l_1\dot{\theta}\sin\theta)^2+(\dot{y}-l_1\dot{\theta}\cos\theta)^2]^{(n_1+1)/2}$$
$$-\frac{b}{n_2+1}[(\dot{x}-l_2\dot{\theta}\sin\theta)^2+(\dot{y}+l_2\dot{\theta}\cos\theta)^2]^{(n_2+1)/2}$$
$$-m_1g\sin\alpha(\dot{y}-l_1\dot{\theta}\cos\theta)$$
$$-m_2g\sin\alpha(\dot{y}+l_2\dot{\theta}\cos\theta).$$

Applying (8), generalized forces corresponding

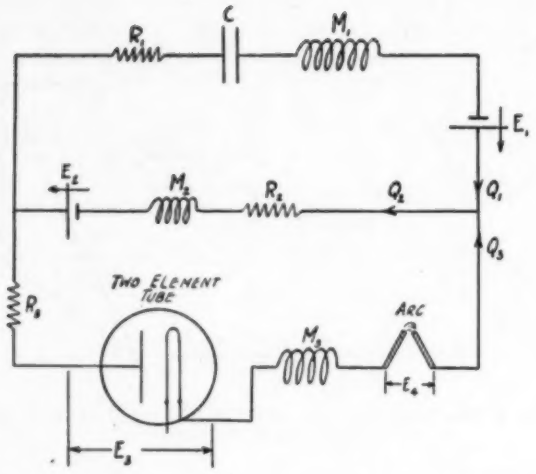

FIG. 2. Electrical system on which dissipative and conservative "forces" are acting.

to x, y and θ can be obtained at once for any values (except -1) of n_1 and n_2. The coefficients "a" and "b" need not be constants but may be functions of coordinates and/or time.

As a special case, suppose that ordinary dynamic friction and gravity are acting. For friction $a=m_1g\mu_1\cos\alpha$, $b=m_2g\mu_2\cos\alpha$, $n_1=n_2=0$, and we find at once that the generalized force corresponding to θ is given by the following rather complicated expression.

$$F_\theta=\frac{-\mu_1m_1g\cos\alpha[l_1(\dot{x}\sin\theta-\dot{y}\cos\theta)+l_1^2\dot{\theta}]}{[(\dot{x}+l_1\dot{\theta}\sin\theta)^2+(\dot{y}-l_1\dot{\theta}\cos\theta)^2]^{\frac{1}{2}}}$$
$$-\frac{\mu_2m_2g\cos\alpha[l_2(\dot{y}\cos\theta-\dot{x}\sin\theta)+l_2^2\dot{\theta}]}{[(\dot{x}-l_2\dot{\theta}\sin\theta)^2+(\dot{y}+l_2\dot{\theta}\cos\theta)^2]^{\frac{1}{2}}}$$
$$+m_1gl_1\sin\alpha\cos\theta-m_2gl_2\sin\alpha\cos\theta.$$

Similar expressions for the generalized forces corresponding to x and y are easily written down.

(2) As a means of illustrating the application of the power function to electrical problems, let us consider the circuit shown in Fig. 2. We shall take as independent "coordinates" the charges Q_1 and Q_3.[3] Thus $\dot{Q}_1=I_1$ is the current in the upper branch of the circuit and $\dot{Q}_3=I_3$ is the current in the lower branch. Let us assume that the voltage across the arc E_4 is given by $E_4=aI_3^{-1}$ where "a" is a constant (this is approximately correct for an open carbon arc) and that E_3, the voltage across the two-element vacuum tube is given by bI_3^n where "b" is a constant and n may have a value between $\frac{1}{2}$ and $\frac{2}{3}$, depending on what part of the voltage-current curve the tube is operating.

Since we are using charge as a coordinate it follows that voltage must play the part of a force. Hence a little consideration will show that the proper power function for this circuit is given by

$$P=-\int\Big[R_1\dot{Q}_1d\dot{Q}_1+R_2(\dot{Q}_1+\dot{Q}_3)(d\dot{Q}_1+d\dot{Q}_3)$$
$$+R_3\dot{Q}_3d\dot{Q}_3+\frac{Q_1}{C}d\dot{Q}_1-E_1d\dot{Q}_1$$
$$-E_2(d\dot{Q}_1+d\dot{Q}_3)+a\dot{Q}_3^{-1}d\dot{Q}_3+b\dot{Q}_3^nd\dot{Q}_3\Big]$$
$$=-\tfrac{1}{2}[R_1\dot{Q}_1^2+R_3\dot{Q}_3^2+R_2(\dot{Q}_1+\dot{Q}_3)^2]-\frac{Q_1}{C}\dot{Q}_1$$
$$+(E_1+E_2)\dot{Q}_1+E_2\dot{Q}_3-a\log\dot{Q}_3-\frac{b}{n+1}\dot{Q}_3^{n+1}.$$

Applying (8), the generalized "force" corresponding to Q_1 is found to be

$$F_{Q_1}=-R_1\dot{Q}_1-R_2(\dot{Q}_1+\dot{Q}_3)-\frac{Q_1}{C}+E_1+E_2.$$

A similar expression for the generalized force corresponding to Q_3 is easily written down.

Combining principles employed in the two examples, generalized forces for electro-mechanical systems can easily be obtained.

CONCLUSIONS

(1) The power function is applicable to a broad field of problems including mechanical, electrical, and electro-mechanical.

(2) A wide variety of forces including dissipative, conservative, externally applied, etc. can in this way be taken care of.

(3) As is evident from the various forms listed and the examples given, the proper power function for a given type of force is easily obtained.

(4) The power function puts the task of determining generalized forces which stem from any or all types of forces mentioned in (2) on the same basis as that of determining conservative forces from a potential-energy function.

[3] For details as to the application of Lagrange's equations to electrical circuits see D. A. Wells, "Application of Lagrange's Equations to Electrical Circuits," J. App. Phys. 9, 312–320 (May 1938).

Flow Orientation in Isotropic Fluids*

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A theory of flow orientation was developed which is based on a relation giving the change in entropy accompanying molecular orientation under the influence of a stress field, and on the applicability of the Eyring theory of viscous flow. The proposed theory predicts the ratio of the viscosity at attainment of complete orientation to the viscosity of "normal" i.e., random flow to be equal to the ratio of the average length $v^{\frac{1}{3}}$ to the actual length of the (fully stretched) molecule. The thermodynamics of a flow-oriented system are outlined, predicting the temperature changes, accompanying adiabatic orientation and disorientation. It is also suggested that the relaxation times in viscous media are large enough to prevent observation of flow orientation in capillary viscosimeters in many cases.

VISCOSITY measurement performed under application of high shear stresses frequently produces "low" values of the viscosity coefficient. This effect is commonly known as "structure viscosity." It has recently been shown[1,2,8,9,12,16,19,21,22] that a large number of such results can be entirely ascribed to heat effects in the rapidly moving liquid. In a much larger number of cases it has, however, not been possible to explain the observed viscosity decreases by heat effects alone. Some other factor had to be considered. The Eyring[6] concept of lowering the potential-energy barrier to flow by the amount of energy supplied by the outside shearing forces according to the equation

$$\eta=\frac{fh\lambda_1}{kT\lambda}\cdot\frac{F}{F_\ddagger}\exp\left[(\epsilon_0-\tfrac{1}{2}f\lambda\lambda_2\lambda_3)/kT\right], \qquad (1)$$

could not account quantitatively for the experimental results, as had already been observed by Eyring and co-workers. The previously proposed hypothesis of molecular-orientation axis parallel with the direction of flow[3,13,23,24] was, therefore, taken up again, and an attempt made to develop a theory of flow orientation in isotropic systems.

A rather complete treatment of flow orientation of anisometric particles in dilute solutions has been presented by Boeder[4] and by Peterlin[18] on hydrodynamic bases. Their theory is, naturally, limited to particles which are large compared with the particles of the solvent and rigid. Many systems of practical interest consist of particles of very nearly uniform size throughout, nor can most of the particles we deal with be thought of as rigid. The following development was not based on either assumption: The theory of viscous flow, as developed by H. Eyring[6] is based on the following model of the liquid state: The "free volume," the commonly accepted criterion of the liquid state, has to form local concentrations (holes) big enough for a molecule to move into. The formation of the hole, and accompanying movements of the molecule in question past its neighbors, requires the passing of potential-energy barriers. The process should, therefore, follow the established laws for the rate of crossing of potential-energy barriers in chemical reactions. A model of the process is given in Fig. 1a. Two layers of molecules in a liquid are at the distance λ_1, A slides past the others under influence of the applied shearing force f. If ΔU be the difference in velocity of the two layers, Newton's law gives

$$\eta\equiv f\lambda_1/\Delta U.$$

The new equilibrium position of A, given by the dotted circle was made possible by the formation of an empty site, a hole. The rate of motion across the potential-energy barrier between these two positions is given by the general-rate equation

$$k_r=(kT/h)\cdot(F/F_\ddagger\exp)\ (-\epsilon_0/kT),$$

or since the height of the potential-energy barrier ϵ_0 is diminished by the applied outside force $\frac{1}{2}f\lambda\lambda_2\lambda_3$ in either direction

$$k_{\text{forward}}=(kT/h)(F_\ddagger/F)\exp\left[-(\epsilon_0-\tfrac{1}{2}f\lambda\lambda_2\lambda_3)/kT\right]=k\exp\left(\tfrac{1}{2}f\lambda\lambda_2\lambda_3/kT\right),$$

$$k_{\text{backward}}=k_r\exp\left(-\tfrac{1}{2}f\lambda\lambda_2\lambda_3/kT\right),$$

giving for the net rate of flow

$$\Delta u=\lambda k_r\sinh\frac{f\lambda\lambda_2\lambda_3}{2kT},$$

resulting for $2kT\gg f\lambda_2\lambda_3\lambda$ in

$$\eta=(\lambda_1 h/\lambda_2\lambda_3\lambda^2)\cdot(F/F_\ddagger)\exp(\epsilon_0/kT). \qquad (2)$$

* Presented before annual meeting of the Society of Rheology, November, 1944.

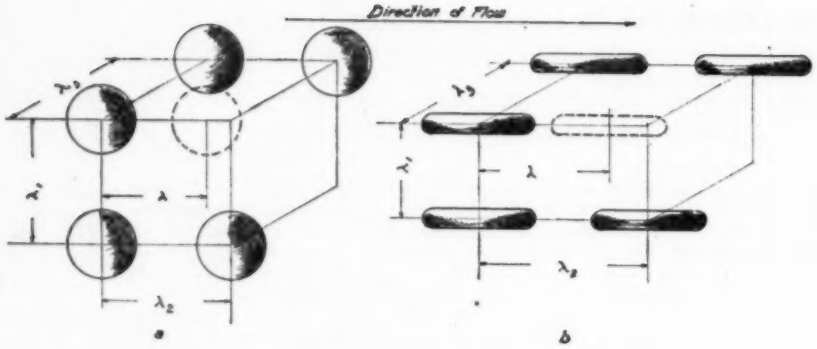

FIG. 1. (a) Eyring's model for the flow process in a system of spheres. (b) Model of system of elongated particles in a state of complete orientation. (For the purposes of clarity the distances between the particles have been greatly exaggerated. Actually the distances λ_i have to be considered almost identical with the respective molecular dimensions, being larger only by an amount $\leq$ than $v_f^{\frac{1}{3}}$.)

In a state of random orientation, the effective molecular distances $\lambda_{1,2,3}$ can all be considered equal regardless of shape and of the order $v^{\frac{1}{3}}$. If, however, the molecules are rod or plate shaped and by some exterior force become completely oriented the quotient $\lambda_1/\lambda_2\lambda_3$ obviously diminishes and with it the viscosity, provided all other factors stay unchanged. λ, the length of the equilibrium jump will—if it is at all affected—increase, thereby further decreasing η. The ratio of the partition functions can be considered unaffected by orientation. ϵ_0 shall here remain unchanged. Inasmuch as we believe the shearing force f to be the orienting force we have now to look for a relation between the degree of orientation and the magnitude of the required shear stress to bring this orientation about, since Eq. (2) does not contain f.

The forces required to orient the members of a "Gaussian" chain have by Guth and James[25] been shown to be a function of the entropy change which accompanies the process of transferring a system from a state of higher probability (complete randomness) to a state of lower probability (orientation). Applying this principle to flow orientation, the forces constraining the oriented molecules from their normal random thermal motion can be identified with the shearing force. Our defining equations can then be written:*

$$S_\omega = k \ln P_{(\omega)}, \tag{3}$$

where $P_{(\omega)}$ = position function of a molecule.

$$Z = -T(\partial S/\partial\varphi)_T = -kT(d/d\varphi) \ln P_{(\omega)}. \tag{4}$$

To solve Eq. (3) and (4) we still have to define $P_{(\omega)}$ and Z.

The normalized position function for the molecule can be written:

$$P_{(\omega)} = \frac{(1-\gamma^2)^{\frac{1}{2}}}{4\pi(1+\gamma\cos 2\varphi)^{\frac{1}{2}}}, \quad \text{where} \quad \gamma = \frac{\lambda_2^2-\lambda_1^2}{\lambda_2^2+\lambda_1^2} \tag{5}$$

* Table of nomenclature at end of article.

and the energy term in analogy to Eyring's procedure:

$$Z = (f\lambda_2^2\lambda/4)\sin\varphi. \tag{6}$$

Substituting Eq. (5) and (6) in (4) and considering that γ is in the cases which are of interest here very close to unity we obtain

$$f = (6kT/\lambda_2^2\lambda)(1/(1-\cos\varphi)). \tag{7}$$

Equation (7) predicts that for $\varphi = 0$ (complete orientation) $f = \infty$, which appears reasonable. For values of f smaller than $6kT/\lambda_2^2\lambda$, however, meaningless negative values for $\cos\varphi$ are obtained. This could well be interpreted to mean that not before $f\lambda_2^2\lambda \geqq 6kT$ does any orientation take place. Considering this, and the fact that the largest portion of f is used to orient the fraction of particles having a larger energy of translation than $f\lambda_2^2\lambda$ we obtain the revised form of (7).

$$f = (6kT/\lambda_2^2\lambda)\exp(-f\lambda_2^2\lambda/kT) \cdot (1/(1-\cos\varphi)-1). \tag{8}$$

From Eq. (8) follows that complete orientation is actually much more readily achieved than Eq. (7) would have led us to expect. It will be seen that there is experimental evidence for this requirement of the theory. Verification of these orientation effects is also theoretically possible by optical means, since the orientation angle φ of Eq. (8) is in a simple manner related to the extinction angle κ, as has been shown by Boeder.[4] The hydrodynamic theory would predict proportionality of Y and sine κ, experimentally verified by Buchheim and Philippoff.[5] The high shear stresses required for the orientation of medium molecular-weight liquids are, however, bound to cause considerable temperature gradients in the moving film. Björnståhl's work[1] suggests that such temperature gradients will produce aberration of the beam of light and reflection inside the instrument. The accuracy of flow-bire-

fringence and extinction angle measurement at high rates of shear will, therefore, be quite low. For the rate of orientation the ordinary rate equation can be expected to hold:

$$1/\tau_\omega = (kT/h) \exp(\Delta S_\ddagger/R) \times \exp(-\Delta H_\ddagger/RT), \quad (9)$$

just as for the relaxation time in the case of electric-dipole orientation.[6] The length of the path of double refracting liquid after exit from the capillary is, so far, the only evidence of a finite value of τ_ω,[25] but precise measurements have not yet been made.

The most important step now is to establish the connection between (8) and the orientation susceptible term $\lambda_1/\lambda_2\lambda_3$ in (2). There λ_2 would have to be replaced by $[(\lambda_2-v^{\frac{1}{3}})\cos\varphi+v^{\frac{1}{3}}]$. In the case of rod shaped molecules λ_1 would be equal to λ_3 and, therefore, both disappear leaving

$$\eta = (h/[(\lambda_2-v^{\frac{1}{3}})\cos\varphi+v^{\frac{1}{3}}]\lambda^2) \cdot (F/F_\ddagger)\exp(\epsilon_0/kT), \quad (10)$$

$$Y_\omega = \eta/\eta_0 = v/[(\lambda_2-v^{\frac{1}{3}})\cos\varphi+v^{\frac{1}{3}}]v^{\frac{2}{3}}, \quad (11)$$

$$1/Y_\omega = 1+\{f/[f+(6kT/\lambda_2{}^2\lambda) \cdot \exp(-f\lambda_2{}^2\lambda/kT)]\}((\lambda_2/v^{\frac{1}{3}})-1). \quad (11')$$

Out of (11) follows that the maximal value for $1/Y_\omega$ at infinite f is equal to $\lambda_2/v^{\frac{1}{3}}$. The reciprocal of this value would constitute the minimum value of Y_ω, the "residual" relative viscosity at complete orientation. The limiting value $Y_{\omega\infty}$ at high shear stresses would therefore constitute a means to determine the effective length of the molecules constituting the liquid under test. Due to the effect of high shear stresses on the degree

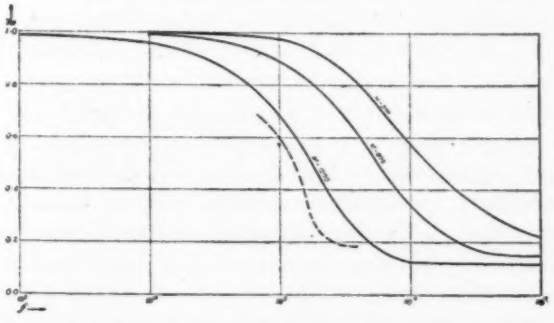

FIG. 2. Relative viscosity change with increasing shear stress due to orientation according to Eq. (11) for hypothetical straight hydrocarbons. (It should not be inferred from these data that the lubricating oil in question may be considered as consisting even partly of straight chain hydrocarbons. All indications are that even the most "paraffinic" petroleum oils contain at least one aromatic or naphthenic ring per molecule. This would affect the axis-ratio, however, but very little, if the alkyl chain is really long, as in the oil under consideration.) Broken line represents experimental results, recalculated from Neale's (reference 24) data corrected for temperature gradient, etc.

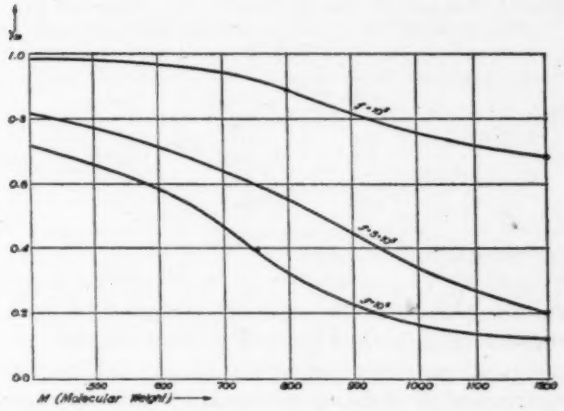

FIG. 3. Dependence of orientation on molecule size according to Eq. (11) for hypothetical straight chain hydrocarbons.

of coiling up of long chain-like molecules, the above equations cannot be expected to hold for very large molecules without further refinement to take care of this "stretching" effect.[10] The amount of ultimate viscosity decrease due to orientation is given by Fig. 2 for straight chains hydrocarbons of up to 1200 molecular weight. Graphs of the Y *vs.* f function for some typical cases are presented on Figs. 2 and 3.

The evaluation of available experimental data is of a somewhat speculative character because the majority of such data have been obtained on lubricating oils rather than pure compounds. The high viscosity component of most of the described automotive lubricants, so called Bright Stock Cylinder oil, has been shown[22] to contain up to 15 percent of compounds with an average molecular weight of 1000 to 1300. The structure of these compounds is not even approximately known. In view of the exponential dependence of orientation on the molecule dimensions a high degree of uncertainty is thus introduced into the calculations. The fair agreement found between theory and experiment might therefore be fortuitous.

It appears in order to compare, at this point, the theory outlined with the hydrodynamic theory of orientation by Boeder and Peterlin. Their theory is based on the following premises: Very dilute suspensions of rigid infinitely thin rods (Boeder) or elongated ellipsoids and platelets (Peterlin) in a medium whose molecules are so small compared to the size of the suspended particles that it can be considered as a continuum. It is also assumed that inertia forces can be neglected. These are essentially the premises introduced by Einstein into the hydro-

TABLE I. Relaxation time of hypothetical elongated particles in liquids of various viscosities calc. acc. to Peterlin.[a]

λ_1(A)	λ_2(A)	η (Poise)	τ (sec.)
10	200	10^4	10
5	150	10^3	10^{-1}
5	75	10^2	10^{-3}
5	50	10	10^{-6}

[a] See reference 18 of Bibliography.

dynamic theory of the specific viscosity of suspensions of spherical particles. The change to elongated particles introduced the additional requirement of defining the angular distribution function of the particles' axes.[11] This position function has therefore been developed as a function of the parameter $\alpha = G/D$ by Boeder and by Peterlin, where: G = the velocity gradient and D the rotational diffusion constant. The latter as formulated by Gans[7] and Perrin[17] on the basis of the Stokes-Einstein relation is a measure of the disorientating effect of thermal motion and contains the dimensions as well as the axis ratio of the particles as material parameters, while the factor G is a measure of the orienting forces. Using Gans' function of D we obtain for α:

$$\alpha = \frac{G}{D} = \frac{G16\eta_0\pi ba^2}{3kT}\frac{1}{[p]}, \tag{12}$$

where

$$[p] = \frac{p^2}{p^4-1}\left[-1+\frac{2p^2-1}{2p(p^2-1)^{\frac{1}{2}}}\ln\frac{p+(p^2-1)^{\frac{1}{2}}}{p-(p^2-1)^{\frac{1}{2}}}\right].$$

Or since $G\eta = f$

$$f = \frac{3kT}{16\pi ba^2\alpha}[p]. \tag{12a}$$

Inasmuch as α and the main orientation angle are related by: $4/\alpha = \text{tang } 2\kappa$ the similarity of (12) with Eq. (8) is quite striking. The resulting expressions for the specific viscosity *vs.* rate of shear function has, of course, no similarity to any of the viscosity functions developed above, expect that they also obtain the axis ratio of the particles from η_∞/η_0. A number of conclusions can, however, be drawn from the hydrodynamic theory which are applicable to any flow-orientation problem.

The time function.—Orientation is not an instantaneous process, but proceeds at a finite rate related to the "relaxation" time of the orientating particle. The classical relaxation time $\tau = 1/6D$. The rate of orientation is thus a function of the particle dimensions, the viscosity of the medium, and the applied rate of shear. Some typical values are given in Table I. For the systems of interest in this investigation the rate is so fast that any time lags observed in the course of viscosity measurements of isotropic fluids have to be ascribed to the rate of attainment of thermal equilibrium of the measuring assembly. Analysis of the figures in Table I points to a fact which has been overlooked so far: This is the influence of relaxation time on the observability of orientation effect. A liquid moving through a capillary of $R = 0.05$ cm and $L = 10$ cm and a rate of shear of 10^5 sec.$^{-1}$ moves from end to end of capillary in 4×10^{-3} sec. A liquid containing particles of $\tau > 10^{-4}$ sec. will, therefore, show practically no orientation effects when measured by capillary flow. It is known from measurements of dipole orientation that relaxation times in concentrated systems are very much (often 100 times) larger than the hydrodynamic theory requires (the interaction effects in dipole orientation, and in flow orientation being most likely identical). This may explain why Hersey and Zimmer could ascribe their results (which were obtained in capillary instruments) to heat effects alone while in the concentric cylinder instrument with longer "contact" time very similar material at the same shear stresses showed decided orientation effects. High polymers in solution will as a rule have such long relaxation times to make axial-ratio determination by high velocity viscometry in capillaries of very questionable value.

Discussion.—A statistical theory of flow orientation is developed in the preceding section which while covering the encountered phenomena fairly completely and also providing numerical results of reasonable agreements with the experiment suffers from a number of serious shortcomings which are essentially inherent in the method of attack. This theory claims as advantage over the hydrodynamic theory to be valid regardless of the concentration of the orientating molecules, but does not provide for any explicit interaction terms. It follows here the lead of the Eyring theory of flow which relegates all interaction terms to the height of the potential energy barrier and the entropy of activation. The success of Eyring's method of approach can for the present be considered as sufficient justification for this simplification. The assumption of independence of the partition-function ratio F/F_i from the degree of orientation is also open to question. Non-spherical but regular distribution

may be expected to accentuate the effect of neglecting the existence of a sinusoidal radial-distribution function in liquids by Eyring's theory. Existent formulations for the normal radial-distribution function in liquids[14] are, however, so uncertain and unwieldy that it did not appear justifiable to consider them in the above derivation.

The analogon to the presented theory of flow orientation is Debye's theory of dipole orientation which can be derived very similarly if the electrical field strength takes the place of the velocity gradient, and the term $\lambda_2^2\lambda\eta$ the place of the dipole moment per molecule. If it were possible to measure the relaxation time of flow-oriented molecules precisely as is possible in the case of electric-dipole orientation, the proper energy and entropy of activation for orientation terms incorporated in Eq. (9) could be independently determined. Such check would be a very good support for the present theory.* Independent of any theory of orientation is, however, the importance of the existence of flow orientation for the energy balance around a flowing liquid. Thermodynamically, the flow of liquids is a strictly irreversible process. The increase in entropy per unit volume can readily be derived from the general equation of motion. Orientation, however, is a reversible process, as has been well proven for the case of magnetic orientation. As in the latter case, the thermodynamic relation in a system subject to flow orientation may be derived as follows:[26] Whenever the degree of orientation of matter changes reversibly as the result of the application of an outside force, the work $-\delta W$ done by the force is:

$$-\delta W = f dI, \tag{15}$$

where I is the "intensity of orientation." It is to be noted that I thus has the dimension of a volume. An interpretation of this observation has not yet been found. Its implicit suggestion to look for volume changes accompanying orientation will, however, be followed up. It is also to be expected that I will be proportionate to $-\ln Y$. We obtain therefore in the course of usual procedure:

$$dE = TdS - PdV + fdI, \tag{16}$$

$$(\partial S/\partial F)_{T,P} = (\partial I/\partial T)_{P,f} = f(\partial\omega/\partial T)_{P,f}, \tag{17}$$

where

$$\omega = I/f,$$

$$S_2 - S_1 = \int_{f_1}^{f_2} f(\partial\omega/\partial T)df = K(f_1^2 - f_2^2), \tag{18}$$

where

$$K \equiv -\frac{6V}{fT}\frac{\exp(-\lambda_2^2\lambda f/kT)}{(\lambda_2/v^1)-1}. \tag{19}$$

It is evident (as $\partial\omega/\partial T$ is negative) that the process of orientation is accompanied by temperature rise, while adiabatic disorientation should be accompanied by heat absorption. If the experiment is conducted under conditions of isothermal flow, the temperature of the liquid in the receiving vessel should be lower than in the capillary.

In comparison with the frictional heat, the enthalpy increase of the liquid due to orientation is very small and of little consequence, in bearing systems. Here, another question is posed: It has been shown by Vogelpohl[22] that the load carrying capacity of a lubricating film is very materially affected by the drop in viscosity due to temperature gradient. Can the same also be said of the decrease in viscosity due to flow orientation? According to the present status of the theory of lubrication, the answer can only be that viscosity decrease, due to flow orientation, should also cause a corresponding decrease in the load carrying capacity of the lubricant film.

In the practice of high speed bearing, lubrication oil is usually fed through the bearing at such a rate that ϑ and therefore ΔY due to temperature gradient are very small. Since the load-carrying capacity of a bearing is proportional to the product $u\cdot\eta$, an increase in speed could safely be accompanied by a corresponding decrease in viscosity. Where every horsepower counts, this viscosity decrease would, in fact, be highly desirable. According to Eq. (11), η is inversely proportional to u in the case of substances showing flow orientation. Lubricants produced from straight-chain compounds would, therefore, answer a definite need in high speed machinery.

To analytical rheology the results of this investigation offer means to correct experimentally observed viscosity values to isothermal as well as "random motion" condition, or to select operating conditions for which these corrections will become negligible. Highly viscous materials of high molecular weight can apparently not be made to flow without causing some degree of flow orientation.[5a] Whether the ("true") vis-

* Such independent check could possibly be provided by determination of the extinction angle and its rate of decay after stopping of the instrument in a further refined model of the instrument for the measurement of flow-birefringence described by J. T. Edsall *et al.* in Rev. Sci. Inst. **15**, 293 (1944).

cosity corrected to zero rate of shear of such materials can be considered a constant of any physical meaning is disputable.

SUMMARY

At very high velocity gradients, the temperature-gradient corrections have been found insufficient to explain the observed decreases in apparent viscosity. It is proposed to explain these phenomena by assumption of flow orientation of the flow elements parallel to the direction of flow. With the method of statistical mechanics and Eyring's theory of flow a theory of flow orientation was evolved which introduces the absolute length and axial ratio of anisodiametrical particles as material constants into the viscosity function. The theory predicts a minimum value of $Y_{\omega\infty}$—the relative "residual" viscosity—at very high rates of shear equal to the ratio of the "average length" to the actual length of a molecule (or flow element).

TABLE OF NOMENCLATURE

D = rotational diffusion constant.
F = partition function (rot. and vitr.).
G = rate of shear (velocity gradient).
f = shear stress.
$H_{\ddagger}$ = standard heat of activation.
h = Planck constant.
k = Boltzmann constant.
p = axial ratio λ_2/λ_1.
S = entropy.
T = abs. temperature.
u = velocity of flow lamina at r.
v = volume per molecule.
x = coordinate normal to direction of flow.
Y = ratio of measured to standard viscosity.
z = coordinate parallel to direction (axis) of flow.
Z = orienting energy.
α = ratio G/D.
γ = parameter in position function $P(\omega)$.
ϵ_0 = height of potential energy barrier to flow.
η = measured viscosity.
η_0 = standard viscosity.
ϑ = temperature difference between apparatus wall and point.
κ = extinction angle.
λ_i = intermolecular distance (λ_1 and λ_{32} = short axis, λ_2 = long axis).
λ = length of equilibrium jump.
ν = kinematic viscosity.
π = constant $\pi = 3.14$.
ρ = density of liquid.
τ = relaxation time.
φ_0 = orientation angle.
ω = "due to orientation."
∞ = residual value of parameter at high shear stresses.
0 = value of parameter at $\sim$ zero shear stress.

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After this paper had gone to print, a paper on flow orientation by J. Frenkel (Acta Physico Chimica, URSS **19**, 51–76 (1944)) has become known through Chem. Abstr. **39**, 1340 (1945). This author derives an orientation distribution function which to a first approximation gives in the nomenclature of this article

$$f = A\frac{\kappa T}{\lambda_1\lambda_2^2}(\cos^2\varphi - \tfrac{1}{3}),$$

where A is a numerical factor close to 1.7. The main difference between this equation and Eq. (8), which have been drived by widely differing methods, is the form of the angle of orientation function. Even this function approaches the one given by us, carrying the solution of Frenkel's integrals to higher approximations.

Three-Dimensional Fourier Transforms and Their Application to Maxwell's Equations

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Boundary value problems in regions with rectangular symmetry can be treated by means of the finite Fourier Transform. Transforms of vector point functions and their divergence and curl are obtained. The transforms of Maxwell's equations take a particularly simple form; the resonator problem is discussed as an example.

INTRODUCTION

THE (ordinary) differential equations of circuit theory are readily solved by the Heaviside calculus or by operational methods based upon the Laplace transform, by means of which differential equations are translated into algebraic equations.[1] E. U. Condon[2] notes that, for engineering problems involving microwaves, no operational method has been developed which avoids the necessity of solving the electromagnetic field equations.

It seems that the three-dimensional finite Fourier Transform (F.T.) provides such a method of treating the electromagnetic field equations in problems with rectangular symmetry. In problems with other symmetry properties (e.g., cylindrical symmetry), the F.T. method reduces the partial differential equations to simpler form. Problems involving transients can be treated by a Laplace transform in time and a F.T. in the space coordinates. We shall, however, discuss only steady-state problems here.

By the F.T. Maxwell's equations between the field quantities are then translated into algebraic equations between vectors.

1. FOURIER TRANSFORMS IN THREE DIMENSIONS

If $Y(x)$ is a function of the real variable x, sectionally continuous and possessing first and second derivatives in the interval $-L<x<L$,[3] then, the finite Fourier transform $y(n)$ of $Y(x)$ is defined as

$$y(n)=F(Y(x))=\int_{-L}^{L}e^{-inx/L}Y(x)dx,$$

where n takes on integral values ($n=0, \pm1, \pm2\cdots$).[4]

The transform of the derivative $Y'(x)$, is then given by:

$$F(Y'(x))=in\pi/LF(Y)+(-1)^n\{Y(L)-Y(-L)\}.$$

Similarly

$$F(Y''(x))=-n^2\pi^2/L^2F(Y)+(-1)^n \times\{in\pi/L(Y(L)-Y(-L))+Y'(L)-Y'(-L)\}.$$

Thus differentiation of the object-function $Y(x)$ corresponds to multiplication of the result-function $y(n)$ by $in\pi/L$ and addition of a term involving only the boundary values of the object function.

This is the main reason why the introduction of the Fourier Transform simplifies the solution of boundary value problems, since differential equations in the object function are reduced to algebraic equations in the result function, which, moreover, contain the boundary conditions.

We can extend the definition of the Fourier transform to cover functions of any number of variables. In particular let $G(x, y, z)$ be a scalar function in the volume V: $(-a_x<x<a_x; -a_y<y<a_y; -a_z<z<a_z)$.

Let us define a vector $\mathbf{n}=\mathbf{i}n_x/a_x+\mathbf{j}n_y/a_y+\mathbf{k}n_z/a_z$ with $n_x, n_y, n_z=0, \pm1, \pm2\cdots$.

We then define $g(n_x, n_y, n_z)$, the three-dimensional Fourier Transform of $G(x, y, z)$ as:

$$g(n_x, n_y, n_z)=F(G(x, y, z))$$

$$=\int\int\int_V \exp[-i\pi(\mathbf{n}\cdot\mathbf{r})]G(x, y, z)dV. \quad (1)$$

The definition for the transform of a vector function $\mathbf{U}(x, y, z)$ is exactly the same, but the result-function $\mathbf{u}(n_x, n_y, n_z)$ will in this case be a

[1] J. L. Barnes and M. F. Gardner, *Transients in Linear Systems* (John Wiley and Sons, Inc., New York, 1942).
[2] E. U. Condon, Rev. Mod. Phys. **14**, 341–89 (1942).
[3] The above assumptions are obviously fulfilled by the electric and magnetic field strengths and in the following we shall always assume that the functions considered satisfy these conditions.
[4] Cf. R. V. Churchill, *Modern Operational Mathematics in Engineering* (McGraw-Hill Book Company, Inc., New York, 1944), Chap. X; and G. Doetsch, Math. Ann. **112**, 52 (1936); H. Kniess, Math. Zeits. **44**, 266 (1939).

vector function of n_x, n_y, and n_z as can be seen by transforming the components of $\mathbf{U}(x, y, z)$.

To derive the transforms of the derivatives of G we must first consider

$$F(\partial G/\partial x) = \iiint_V \exp[-i\pi(\mathbf{n}\cdot\mathbf{r})]\partial G/\partial x dx dy dz.$$

After an integration by parts this becomes:

$$F(\partial G/\partial x) = \iint \exp[-i\pi(\mathbf{n}\cdot\mathbf{r})]G dy dz\Big|_{-a_x}^{a_x} + i\pi n_x/a_x \iiint \exp[-i(\mathbf{n}\cdot\mathbf{r})]G dx dy dz.$$

The first term on the right is the difference between two surface integrals, one taken over the plane $x=a_x$, the other over the plane $x=-a_x$. Denoting this difference by $f_x(G)$, we get

$$F(\partial G/\partial x) = i\pi n_x/a_x F(G) + f_x(G). \tag{i}$$

In the same way we obtain

$$F(\partial G/\partial y) = i\pi n_y/a_y F(G) + f_y(G). \tag{ii}$$

$$F(\partial G/\partial z) = i\pi n_z/a_z F(G) + f_z(G). \tag{iii}$$

The gradient of G is given by the formula $\text{grad } G = \mathbf{i}\partial G/\partial x + \mathbf{j}\partial G/\partial y + \mathbf{k}\partial G/\partial z$. Thus, adding (i), (ii), and (iii) vectorially, we get:

$$\mathbf{F}(\text{grad } G) = i\pi\mathbf{n}F(G) + \mathbf{i}\iint \exp[-i\pi(\mathbf{n}\cdot\mathbf{r})]G dy dz\Big|_{-a_x}^{a_x} + \mathbf{j}\iint \exp[-i\pi(\mathbf{n}\cdot\mathbf{r})]G dx dz\Big|_{a_y}^{a_y} + \mathbf{k}\iint \exp[-i\pi(\mathbf{n}\cdot\mathbf{r})]G dx dy\Big|_{-a_z}^{a_z}.$$

The last three integrals now add up to a vector integral over the entire surface bounding V, and thus we get:

$$\mathbf{F}(\text{grad } G) = i\pi\mathbf{n}F(G) + \iint_S \exp[-i\pi(\mathbf{n}\cdot\mathbf{r})]G d\mathbf{S}, \tag{2}$$

where $\mathbf{dS}$ has the direction of the vector normal to the surface S.

If $\mathbf{U}$ is a vector function $\mathbf{U} = \mathbf{i}X + \mathbf{j}Y + \mathbf{k}Z$, the Fourier Transform of $\mathbf{U}$ will be a vector, and it will be denoted by

$$\mathbf{F}(\mathbf{U}) = \iiint \exp[-i\pi(\mathbf{n}\cdot\mathbf{r})]\mathbf{U} dV.$$

The divergence of $\mathbf{U}$ is given by: $\text{div } \mathbf{U} = \partial X/\partial x + \partial Y/\partial y + \partial X/\partial z$, and substituting X, Y, Z for G in (i), (ii), and (iii) respectively and adding, we get:

$$F(\text{div } \mathbf{U}) = i\pi(\mathbf{n}\cdot\mathbf{F}(\mathbf{U})) + \iint_S \exp[-i\pi(\mathbf{n}\cdot\mathbf{r})](\mathbf{U}\cdot d\mathbf{S}). \tag{3}$$

For the curl of $\mathbf{U}$:

$$\begin{aligned}\text{curl } \mathbf{U} = {} & \mathbf{i}(\partial Z/\partial y - \partial Y/\partial z) \\ & + \mathbf{j}(\partial X/\partial z - \partial Z/\partial x) \\ & + \mathbf{k}(\partial Y/\partial x - \partial X/\partial y),\end{aligned}$$

we obtain, substituting again into (i), (ii), and (iii)

$$\begin{aligned}\mathbf{F}(\text{curl } \mathbf{U}) = {} & \mathbf{i}(i\pi n_y/a_y F(Z) - i\pi n_z/a_z F(Y)) \\ & + \mathbf{j}(i\pi n_z/a_z F(X) - i\pi n_x/a_x F(Z)) \\ & + \mathbf{k}(i\pi n_x/a_x F(Y) - i\pi n_y/a_y F(X)) \\ & + \mathbf{i}(f_y(Z) - f_z(Y)) + \mathbf{j}(f_z(X) - f_x(Z)) + \mathbf{k}(f_x(Y) - f_y(X)).\end{aligned}$$

This reduces to:

$$\mathbf{F}(\text{curl } \mathbf{U}) = i\pi[\mathbf{n}\times\mathbf{F}(\mathbf{U})] + \iint_S \exp[-i\pi(\mathbf{n}\cdot\mathbf{r})][d\mathbf{S}\times\mathbf{U}]. \tag{4}$$

The transforms (2)–(4) can also be obtained as follows: We note that grad $(Gv) = G \text{ grad } v + v \text{ grad } G$, and thus:

$$\mathbf{F}(\text{grad } G) = \int\int\int_V \exp[-i\pi(\mathbf{n}\cdot\mathbf{r})] \text{ grad } G dV$$

$$= \int\int\int_V \text{grad } (\exp[-i\pi(\mathbf{n}\cdot\mathbf{r})]G) dV + i\pi\mathbf{n}\int\int\int_V \exp[-i\pi(\mathbf{n}\cdot\mathbf{r})]G dV.$$

The first integral on the right transforms according to a theorem on volume integrals[5] into the surface integral

$$\int\int_S \exp[-i\pi(\mathbf{n}\cdot\mathbf{r})]G d\mathbf{S},$$

over the surface S bounding the volume V, and the second integral on the right is simply $F(G)$, so that we have again (2).

If $\mathbf{U}(x, y, z)$ is a vector function we can derive the formulas for $F(\text{div } \mathbf{U})$ and $\mathbf{F}(\text{curl } \mathbf{U})$ in an analogous fashion: div $(\mathbf{U}v) = v \text{ div } \mathbf{U} + (\mathbf{U}\cdot\text{grad } v)$, and thus

$$F(\text{div } \mathbf{U}) = \int\int\int_V \exp[-i\pi(\mathbf{n}\cdot\mathbf{r})] \text{ div } \mathbf{U} dv$$

$$= \int\int\int_V \text{div } (\exp[-i\pi(\mathbf{n}\cdot\mathbf{r})]\mathbf{U}) dv + i\pi(\mathbf{n}\cdot\int\int\int_V \exp[-i\pi(\mathbf{n}\cdot\mathbf{r})]\mathbf{U} dV).$$

The first integral on the right again transforms into a surface integral; according to Gauss' theorem it is equivalent to

$$\int\int_S \exp[-i\pi(\mathbf{n}\cdot\mathbf{r})](\mathbf{U}\cdot d\mathbf{S}),$$

and thus we get (3).

Similarly, noting that curl $(\mathbf{U}v) = v \text{ curl } \mathbf{U} + [\mathbf{U}\times\text{grad } v]$, and transforming according to Stokes' theorem, we get (4).

From (2) and (3) we can obtain the formula for the transform of the Laplacian of a scalar function $G(x, y, z)$ by observing that $\nabla^2 G = \text{div grad } G$. Thus, by substituting grad G for $\mathbf{U}$ in (3) and transforming the first term on the right by (2), we obtain:

$$F(\nabla^2 G) = -\pi^2\mathbf{n}^2 F(G) + i\pi(\mathbf{n}\cdot\int\int_S \exp[-i\pi(\mathbf{n}\cdot\mathbf{r})]G d\mathbf{S}) + \int\int_S \exp[-i\pi(\mathbf{n}\cdot\mathbf{r})](\text{grad } G\cdot dS). \quad (5)$$

If we write the symbol $\mathbf{f}_{(s)}$ for the integral vector operator

$$\int\int_S \exp[-i\pi(\mathbf{n}\cdot\mathbf{r})](\quad) d\mathbf{S},$$

we can write Eqs. (2)–(5) in the simple form:

$$\mathbf{F}(\text{grad } G) = i\pi\mathbf{n}F(G) + \mathbf{f}_{(s)}G, \quad (2a)$$

$$F(\text{div } \mathbf{U}) = i\pi(\mathbf{n}\cdot\mathbf{F}(\mathbf{U})) + (\mathbf{f}_{(s)}\cdot\mathbf{U}), \quad (3a)$$

$$\mathbf{F}(\text{curl } \mathbf{U}) = i\pi[\mathbf{n}\times\mathbf{F}(\mathbf{U})] + [\mathbf{f}_{(s)}\times\mathbf{U}], \quad (4a)$$

$$F(\nabla^2 G) = -\pi^2 n^2 F(G) + i\pi(\mathbf{n}\cdot\mathbf{f}_{(s)}G) + (\mathbf{f}_{(s)}\cdot\text{grad } G). \quad (5a)$$

In the one-dimensional case differentiation of the object function corresponds to multiplication of the result function by $i\pi n$. We see that in the three-dimensional case the differential operations

[5] J. W. Gibbs and E. B. Wilson, *Vector Analysis* (Yale University Press, New Haven, 1922), p. 409.

$\bar{\nabla}G$, $\bar{\nabla}\cdot\mathbf{U}$ and $[\bar{\nabla}\times\mathbf{U}]$ are translated by the F.T. into $i\pi\mathbf{n}F(G)$, $i\pi(\mathbf{n}\cdot\mathbf{F}(\mathbf{U}))$, and $i\pi[\mathbf{n}\times\mathbf{F}(\mathbf{U})]$, respectively. While in the one-dimensional case the boundary values of the object function occur explicitly in the transforms of the derivatives, they enter here through the surface integrals $\mathbf{f}_{(s)}G$, $(\mathbf{f}_{(s)}\cdot U)$, etc., and, again, just as in the one-dimensional case, the transform of the second derivative (i.e., of the Laplacian) depends on the boundary values not only of the function itself, but also of its first derivative (i.e., of the gradient).[6]

2. MAXWELL'S EQUATIONS IN A HOMOGENEOUS MEDIUM

The introduction of the three-dimensional finite Fourier Transform simplifies the treatment of certain linear partial differential equations of physics in Cartesian coordinates.

For instance, Maxwell's equations, in a medium of constant magnetic permeability μ and constant dielectric constant ϵ are:

$$\begin{aligned}&\epsilon\ \mathrm{div}\ \mathbf{E}=4\pi\rho,\quad \mu\ \mathrm{div}\ \mathbf{H}=0,\\ &\mathrm{curl}\ \mathbf{E}=-\mu/c\partial\mathbf{H}/\partial t,\\ &\mathrm{curl}\ \mathbf{H}=4\pi\mathbf{I}+\epsilon/c\partial\mathbf{E}/\partial t,\end{aligned}\qquad (A)$$

where $\rho(x, y, z)$ is the charge density and $\mathbf{I}(x, y, z)$ the current density. Differentiation with respect to a parameter (in the present case the time, t) is interchangeable with the operator $F(\ )$. Hence, using Eqs. (3a) and (4a) to obtain the transforms of the vector derivatives, we get from (A):

$$i\pi\epsilon(\mathbf{n}\cdot\mathbf{F}(\mathbf{E}))=-(\mathbf{f}_{(s)}\cdot\mathbf{E})+4\pi F(\rho),\qquad (6)$$

$$i\pi\mu(\mathbf{n}\cdot\mathbf{F}(\mathbf{H}))=-(\mathbf{f}_{(s)}\cdot\mathbf{H}),\qquad (7)$$

$$i\pi[\mathbf{n}\times\mathbf{F}(\mathbf{E})]=-[\mathbf{f}_{(s)}\times\mathbf{E}]-\mu/cd/dt(\mathbf{F}(\mathbf{H})),\qquad (8)$$

$$i\pi[\mathbf{n}\times\mathbf{F}(\mathbf{H})]=-[\mathbf{f}_{(s)}\times\mathbf{H}]+4\pi\mathbf{F}(\mathbf{I})+\epsilon/cd/dt(\mathbf{F}(\mathbf{E})).\qquad (9)$$

3. RECTANGULAR CAVITY RESONATOR WITH PERFECTLY CONDUCTING WALLS

As an example we shall consider the problem of the cavity resonator with perfectly conducting walls. In this case we have to solve Eqs. (6)–(9) under the conditions $\rho=0$, $\mathbf{I}=0$, and, at the boundaries $\mathbf{E}$ everywhere normal to the walls and $\mathbf{H}$ everywhere tangential to the walls. From the last two conditions it follows that $(\mathbf{f}_{(s)}\cdot\mathbf{H})=0$ and $[\mathbf{f}_{(s)}\times\mathbf{E}]=0$, since the vector operator $\mathbf{f}_{(s)}$ has everywhere the direction of the vector normal to the surface. However, the conditions for $\mathbf{E}$ and $\mathbf{H}$ hold for each point of the surface, and the $\mathbf{E}$ and $\mathbf{H}$ obtained from the transform equations will contain constants which will have to be determined by the conditions $\mathbf{E}$ tangential and $\mathbf{H}$ normal to the surface themselves.

[6] The appearance of these spurious boundary values of the function which are not given by the problem is considered by Doetsch (reference 3).

For the solution, by the transform method, of eigenvalue problems, like the one we are considering, it is desirable to have all the boundary value conditions drop out of the equations. This can be done very easily in the present case by adjusting the constants a_x, a_y, and a_z entering into the vector $\mathbf{n}$ in such a way that the resonator has the dimensions

$$0<x<a_x,\quad 0<y<a_y,\quad 0<z<a_z.$$

We can then continue the vector functions $\mathbf{E}$ and $\mathbf{H}$ as even functions of x, y, and z over the remainder of the volume V. In this case all the surface integrals over the surfaces bounding the volume V will vanish, and the Fourier Transforms of Maxwell's equations become:

$$(\mathbf{n}\cdot\mathbf{F}(\mathbf{E}))=0,\qquad (6a)$$

$$(\mathbf{n}\cdot\mathbf{F}(\mathbf{H}))=0,\qquad (7a)$$

$$i\pi[\mathbf{n}\times\mathbf{F}(\mathbf{E})]+\mu/cd/dt(\mathbf{F}(\mathbf{H}))=0,\qquad (8a)$$

$$i\pi[\mathbf{n}\times\mathbf{F}(\mathbf{H})]-\epsilon/cd/dt(\mathbf{F}(\mathbf{E}))=0.\qquad (9a)$$

From (6a) and (7a) we see that $\mathbf{F}(\mathbf{E})$ and $\mathbf{F}(\mathbf{H})$ are perpendicular to $\mathbf{n}$. We can thus write, using (7a):

$$\mathbf{F}(\mathbf{H})=[\mathbf{n}\times\mathbf{a}(\mathbf{n})]\quad\text{with}\quad(\mathbf{a}(\mathbf{n})\cdot\mathbf{n})=0\qquad (10)$$

where the vector $\mathbf{a}$ is a function of $\mathbf{n}$ and t. As will be seen, $\mathbf{a}$ is essentially a resonator coordinate.

Substituting (10) into (8a) we get:

$$[\mathbf{n}\times\{i\pi\mathbf{F}(\mathbf{E})+\mu/cd\mathbf{a}/dt\}]=0,$$

and thus:

$$i\pi\mathbf{F}(\mathbf{E})+\mu/cd\mathbf{a}/dt=\varphi(\mathbf{n})\mathbf{n}.\qquad (11)$$[7]

Since the scalar products of $\mathbf{F}(E)$ and $\mathbf{a}$ with $\mathbf{n}$ vanish according to (6a) and (10), we see that for the case we are considering $\varphi(\mathbf{n})=0$, and we have

$$\mathbf{F}(\mathbf{E})=i\mu/\pi cd\mathbf{a}/dt.\qquad (11a)$$

Substituting back for $\mathbf{F}(E)$ and $\mathbf{F}(H)$ into (9a)

[7] (10) and (11) are the transforms of the equations giving $\mathbf{E}$ and $\mathbf{H}$ in terms of the vector potential $\mathbf{A}$ and the scalar potential φ and hold quite generally, since we have not yet used the fact that $\mathrm{div}\ \mathbf{E}=0$.

we get:

$$i\pi[\mathbf{n}\times[\mathbf{n}\times\mathbf{a}]]-i\mu\epsilon/\pi c^2 d^2\mathbf{a}/dt^2=0,$$

which, because of $(\mathbf{a}\cdot\mathbf{n})=0$, becomes:

$$\mathbf{n}^2\mathbf{a}+\mu\epsilon/\pi^2c^2d^2\mathbf{a}/dt^2=0. \qquad (12)$$

Taking the steady-state solution of (12) in the form $\mathbf{a}=\mathbf{a}_0e^{i\omega t}$ we have

$$(\mathbf{n}^2-\omega^2\mu\epsilon/\pi^2c^2)\mathbf{a}=0. \qquad (13)$$

Here $\mathbf{a}=\mathbf{a}(\mathbf{n}, t)$, and (13) must hold for all values of $\mathbf{n}$. This is possible only if either $\mathbf{a}$ is identically zero for all $\mathbf{n}$, in which case the F.T. of both $\mathbf{E}$ and $\mathbf{H}$,—and thus the field quantities themselves—are identically zero; or if ω is such that there exists one or more particular values n_l of n so that $\omega^2=\mathbf{n}_l{}^2\pi^2c^2/\mu\epsilon$, where we recall that

$$\mathbf{n}^2=(n_x/a_x)^2+(n_y/a_y)^2+(n_z/a_z)^2 \qquad (n_x, n_y, n_z=0, \pm1, \pm2, \cdots).$$

Thus the proper frequencies for the resonator follow from (13).

For any particular resonant frequency ω there may be several modes of oscillation, since a value of n^2 might result from more than one set of values $n_x{}^{(l)}, n_y{}^{(l)}, n_z{}^{(l)}$. As pointed out by Condon,[8] this latter case will occur if there is degeneracy because of the symmetry of the resonator.

To derive the functional form of the field vectors note that for any particular mode l of oscillation associated with the frequency ω we have

$$\mathbf{a}(\mathbf{n})=0, \qquad \mathbf{n}\neq\mathbf{n}_l,$$

$$\mathbf{a}(\mathbf{n})=\text{arbitrary}, \qquad \mathbf{n}=\mathbf{n}_l.$$

That is $\mathbf{a}(\mathbf{n})=\mathbf{a}_l{}^0\delta_{\mathbf{n},\mathbf{n}_l}$, where $\mathbf{a}_l{}^0$ is parallel to $\mathbf{a}(\mathbf{n}_l)$, that is from (10) perpendicular to $\mathbf{n}_l$, and arbitrary in magnitude.[9] From (11a) we get:

$$\mathbf{F}(\mathbf{E})=-\frac{\mu\omega}{\pi c}\mathbf{a}(\mathbf{n}). \qquad (14)$$

It can be seen by inspection that this $F(E)$ corresponds to the object function:

$$\mathbf{E}=-\frac{\mu\omega}{\pi c}\mathbf{a}_l{}^0\exp[i(\mathbf{n}_l\cdot\mathbf{r})+i\omega t].$$

The field vector will be the real part of this, with the sine-cosine alternatives specialized to fulfill the boundary condition that $\mathbf{E}$ must be everywhere perpendicular to the boundary of V.

Noting that

$$\mathbf{n}=\mathbf{i}n_x{}^{(l)}/a_x+\mathbf{j}n_y{}^{(l)}/a_y+\mathbf{k}n_z{}^{(l)}/a_z,$$

we get with $C=-(\mu\omega/\pi c)$

$$\begin{aligned}
E_x&=Ca_x{}^0\cos(\pi x n_x{}^{(l)}/a_x)\\
&\quad\times\sin(\pi y n_y{}^{(l)}/a_y)\sin(\pi z n_z{}^{(l)}/a_z),\\
E_y&=Ca_y{}^0\sin(\pi x n_x{}^{(l)}/a_x)\\
&\quad\times\cos(\pi y n_y{}^{(l)}/a_y)\sin(\pi z n_z{}^{(l)}/a_z),\\
E_z&=Ca_z{}^0\sin(\pi x n_x{}^{(l)}/a_x)\\
&\quad\times\sin(\pi y n_y{}^{(l)}/a_y)\cos(\pi z n_z{}^{(l)}/a_z).
\end{aligned} \qquad (15)$$

The formula for $\mathbf{H}$ is obtained in exactly the same way by observing that, from (10)

$$\mathbf{F}(\mathbf{H})=[\mathbf{n}\times\mathbf{a}],$$

and thus:

$$\mathbf{H}=C\left(\frac{\epsilon}{\mu}\right)^{\frac{1}{2}}[\mathbf{n}_l\times\mathbf{a}_l{}^0]\exp[i\pi(\mathbf{n}_l\cdot\mathbf{r})+i\omega t]. \qquad (16)$$

If we want to obtain E in the case of degeneracy where several modes belonging to the same ω are excited, we get from (14) and (11a):

$$\mathbf{F}(\mathbf{E})=-\frac{\mu\omega}{\pi c}\sum_{l=1}^{k}a_l{}^0\delta_{\mathbf{n}_l,\mathbf{n}}e^{i\omega t},$$

which inverts to a sum of the $\mathbf{E}$ belonging to the different modes.

From (16) we see at once that $\mathbf{H}$ is perpendicular to $\mathbf{n}_k$. $\mathbf{E}$ is also perpendicular to $\mathbf{n}_k$, since, from (10), $(\mathbf{n}\cdot\mathbf{a})=0$. Now $\mathbf{n}_k$ is the wave vector, since $E=C\mathbf{a}_0\exp[i\pi(\mathbf{n}_k\cdot\mathbf{r})+i\omega t]$. Thus we see that both $\mathbf{E}$ and $\mathbf{H}$ are perpendicular to their direction of propagation as well as to each other.

In a medium of finite ρ' resistivity ρ' we have $\mathbf{E}=$ constant I, and Eq. (9a) would become

$$i\pi[\mathbf{n}\times\mathbf{F}(\mathbf{H})]-\epsilon/cd/dt(\mathbf{F}(\mathbf{E}))=+(4\pi/\rho')\mathbf{F}(\mathbf{E}), \qquad (17)$$

while Eqs. (6a) to (8a) would remain unchanged. The method used above is applicable in this case also.

Equations (6a)–(12) derived here by the finite Fourier transform method remain formally unchanged if one or more of the dimensions a_x, a_y, and a_z of the region considered are allowed to

[8] E. U. Condon, reference 2, p. 348.

[9] $\delta_{\mathbf{n}\mathbf{n}_l}$ is defined to be equal to one for $\mathbf{n}=\mathbf{n}_l$ and zero otherwise.

increase without limit. If a_x, a_y, and a_z go to infinity the finite three-dimensional Fourier Transform becomes a three-dimensional Fourier Integral Transform; the functions **G** and **U**, respectively **E** and **H** have then to be restricted in such a way that they satisfy the conditions for the existence of a Fourier Integral Transform.

If, say a_x and a_y remain finite while a_z goes to infinity, the form of the equations for the transforms remain the same but it has to be remembered that **n** becomes

$$\mathbf{n}=(n_x/a_x)\mathbf{i}+(n_y/a_y)\mathbf{j}+n_3\mathbf{k},$$

where

$$n_x, n_y=0, \pm 1; \pm 2; \cdots,$$

and n_3 can take any real value; the inversion with respect to n_x and n_y is then a finite Fourier Transform inversion while the inversion with respect to n_3 is a Fourier integral inversion. Problems of this type will be considered elsewhere.

New Booklets

Nickel Steel Topics for June, 1945 features stories on a 13-mile tunnel bored under the Rockies for an irrigation project, a huge dragline bucket built for coal stripping in Pennsylvania anthracite fields, and the use of nickel alloy steel in the nose frame of the B-29. 8 pages. The International Nickel Company, Inc., 67 Wall Street, New York 5, New York.

RCA Victor Division of Radio Corporation of America, Camden, New Jersey, is presenting an informative panorama of the increasingly varied uses of electronic sound systems in industry, institutions, and commercial organizations in a new illustrated brochure entitled *RCA Sound Systems*. A distinctive feature of the brochure is the use of block diagrams to present graphically the special services rendered by sound in various types of establishments and the arrangement of control consoles, microphones, and loudspeakers in different kinds of installations. The needs of industrial plants and offices, schools, hospitals, and penal institutions, and various types of commercial establishments are covered by the diagrams and accompanying text.

In announcing its change in name to American Optical Company, Scientific Instrument Division, Spencer Lens Company, Buffalo 11, New York, has published a 77-page booklet, illustrated with numerous engravings, entitled *Three American Microscope Builders*. It recounts the achievements of Charles A. Spencer, Robert T. Tolles, and Herbert R. Spencer in the early days of the development of the microscope.

All American Tool and Manufacturing Company, 1014 Fullerton Avenue, Chicago 14, Illinois, has published its Bulletin 425 entitled *Post-War Competition Demands Rigid Product Testing*, which discusses the ways in which vibration fatigue testing of post-war products will insure their survival during and after the highly competitive post-war period. The bulletin describes the company's line of "shakedown" test equipment for research and development laboratories as well as for inspection and testing departments. Typical manufacturers, technical schools, and governmental departments using these machines are also listed. 4 pages.

The Milway Notebook, a 34-page booklet containing many feature stories on scientific developments, is published quarterly and distributed free to industries, schools, and others interested in laboratory and research work. Send a request on your letterhead, being sure to give your complete address and zone number, to Chicago Apparatus Company, 1735 North Ashland Avenue, Chicago 22, Illinois.

Lindberg Engineering Company, 2450 West Hubbard Street, Chicago 12, Illinois, has published three new Bulletins: No. 160, *Gas Fired Toolroom Box Type Furnace*, No. 170, *Gas Fired Production Box Type Furnace*, and No. 201, *All Purpose Brazing Furnace*.

Leeds and Northrup Company has published a new 36-page edition of its catalogue, *Apparatus for D-C Resistance Measurements*. This catalogue presents d.c. resistance bridges and also the standards, galvanometers, and accessories usually chosen for use with them. Instruments for a wide variety of applications are described, including models for the most precise measurements, as well as those designed especially for routine testing, student instruction, and many others. Each instrument is so listed and described that readers can easily choose the items best suited for their specific use. Write for Catalogue E-53, Leeds and Northrup Company, 4934 Stenton Avenue, Philadelphia 44, Pennsylvania.

"The Penetron," exclusively manufactured and sold by Engineering Laboratories, Inc., Tulsa, Oklahoma, is discussed in an 18-page booklet published by Texaco Development Corporation, New York, New York. The government has recently lifted secrecy from this device, which utilizes electronic and radioactivity principles for the measurement of thickness of solids, metallic or non-metallic, and for determining liquid level and liquid density. By means of The Penetron accurate inspection data are obtained on high pressure pipe lines, boiler tubes, towers, pressure vessels, etc., without cutting or drilling. Required annual inspections formerly cost an average of $10,000 per location and required complete shutdown of equipment. By use of The Penetron accurate readings are made in 25 seconds, with over 150 inspections per working day possible.

New Book

A Shorter History of Science

By Sir William Cecil Dampier. Pp. 189, Figs. 14, Plates IX, 20×13 cm. The Macmillan Company, New York, 1944. Price $2.00

This interesting little book will be particularly helpful for those who are anxious for a quick review of the history of science. As the author states in his preface, it will be found especially useful to the general reader who wishes to trace in an afternoon's reading the history of science from its prehistoric origins through to the recent developments in physics, chemistry, and astronomy. The book is well illustrated with drawings and photographs.

Erratum: On the Representation of Rigid Rotations

E. T. Benedikt

[J. App. Phys. 15, 614 (Aug. 1944)]

MR. Peter T. Landsberg of London has pointed out a number of misprints in the article as titled above. The equations should be amended to read as follows:

Eq. (1): $z'=Z-2\sin^2\frac{\theta}{2}(z\sin^2\gamma-x\cos\gamma\cos\alpha-y\cos\gamma\cos\beta)+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}(z\cos\alpha-x\cos\beta).$

Eq. (1′): $$\mathbf{r}'=\mathbf{r}-2\sin^2\frac{\theta}{2}[\mathbf{r}-\mathbf{r}\cdot\mathbf{h}\,\mathbf{h}]+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}[\mathbf{h}\times\mathbf{r}].$$

Eq. (7): $$(\mathbf{v}\times)^2\mathbf{y}=[\mathbf{v}\cdot\mathbf{y}]\mathbf{v}-v^2\mathbf{y}=[(\mathbf{v}^2)-v^2]\mathbf{y}.$$

Eq. (8): $$(\mathbf{v}\times)^{2m}=(\mathbf{v}\times)^2(\mathbf{v}\times)^{2m-2}=(-1)^{m-1}v^{2m-4}[(\mathbf{v}^2)-v^2]^2.$$

Eq. (10): $$\mathfrak{R}\equiv 1+\sum_{s=1}^{\infty}\frac{(-1)^{s+1}}{(2s)!}\theta^{2s-2}[\boldsymbol{\theta}^2-\theta^2]+\sum_{s=0}^{\infty}\frac{(-1)^s}{(2s+1)!}\theta^{2s}\boldsymbol{\theta}\times=\sum_{s=0}^{\infty}\frac{1}{s!}(\boldsymbol{\theta}\times)^s=\exp\boldsymbol{\theta}\mathbf{h}\times.$$

Letter to the Editor

Arthur M. G. Moody
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IN the May issue of the *Journal* there appears a paper entitled "A Novel Form of Refrigerator" by J. R. Roebuck, which calls for a good deal of comment. Professor Roebuck appears to have done very little investigation into previous work in this field, and his analysis and design solution of the problem contain serious flaws. The author, furthermore, by implication asks some questions to which the writer would like to suggest some answers.

First of all, let us consider the "novelty" which is implied in the title. One is struck by the total lack of reference to Nernst, who proposed this type of machine long ago as a gas turbine. Surprisingly enough, Professor Roebuck has a reference to Stodola's book "Steam and Gas Turbines," on page 1220 of which is described Nernst's machine, along with an analysis of the thermodynamics. There is, furthermore, an extensive patent art which I suspect would reveal a good deal of similar material. Kapitza, for instance, holds an American patent dealing with the cooling of gases by expanding them in a work-producing machine. The author moreover refers to the use of the reversed refrigerating cycle for domestic heating without indicating any awareness of the tremendous amount of work being carried on in this field by industry today.

Second, let us discuss some of the details of mechanical design. Professor Roebuck apparently would like a little guidance on feasible operating speeds. He attempts to indicate an answer by quoting figures as to permissible peripheral speeds. These, however, are taken completely out of context. Stodola's reference to 1300 f.p.s. applies to disk wheels carrying a peripheral ring of turbine blades. He wrote the book in 1925, and there has been improvement to perhaps 1450 or 1500 f.p.s. since that date. Beams' figure refers to centrifuge rotos, which are structurally better able to run fast, and the figure quoted is not out of the question. For the author's machine, we have a very awkward type of construction (which unfortunately may be inherent in the cycle) which cannot be considered for high speeds. We can, for instance, use as a limiting value the free ring stress encountered at the speeds in question, since the machine is full of more or less unsupported rotating rings. Assuming that we are going to be bold, we might adopt a stress of 50,000 p.s.i. for a very good chrome-nickel steel; we then find our peripheral speed limited to 690 f.p.s. By the time we introduce the dead load of the septa, the cooling water, and possibly part of the load of the perforated cylinders, we shall do well to operate at 500 f.p.s. (and I mean amazingly well). The energy extraction then becomes about 2.8 calories per gram, or less than 12° refrigeration. A detailed stress analysis might indicate more favorable mechanical performance, but the writer believes this to be most unlikely.

Third, let us consider the thermal and hydraulic performance. The paper completely ignores losses. This is permissible in a preliminary study, but this paper goes a good deal further, as, for instance, in the observation which begins: "Since this rotor cooler appears to be the most efficient refrigerator available." Losses there will be, and they will be appreciable. There will not be perfect heat transfer. The ultimate performance will probably be inferior and possibly far inferior to that of machines now being built commercially.

We come now to some points in the author's analysis which display a misconception of what goes on in a heat

engine or pump. For example, he states that in all engines other than this one "the falling of the pressure due to falling gas temperature lessens markedly the work extracted by the piston." This is either meaningless or false, I am not sure which. An adiabatic expansion is an adiabatic expansion, regardless of the type of machine in which it occurs, whether a cylinder with a piston, a turbine nozzle, or the tube of Professor Roebuck's (or Nernst's) device. The remark quoted is just silly.

There is something rather mystical in the "situation, where a definite force can be exerted on each molecule of the working fluid without the immediate presence of solid bodies to accomplish it." This machine has plenty of solid bodies, and if it did not, it simply would not work. As for accomplishing heat transfer to and from a gas without interfering with work relations in the gas, that is done in machines built by the writer's company as long ago as seven years, but we should not have dreamed of calling it unique. This matter is briefly touched on in a paper, "High Temperature Gas Turbines," by J. S. Haverstick and A. M. G. Moody, *Mechanical Engineering*, April, 1945.

In summary, there is a large amount of previous work in this field, particularly in the patent literature, to which no reference has been made. Because of the difficulty of knowing how much promise a device of the kind Professor Roebuck describes has, it is suggested that a model should be built and tested. If the design is carried thus far, the assistance of an experienced mechanical designer would be very useful.

Here and There

New Appointments

The Blakiston Company, publisher in many science fields, has announced the appointment of James P. Lackey as editor of undergraduate subjects in the sciences.

Dr. George E. Ziegler, physicist on the staff of the Midwest Research Institute, has been appointed executive scientist in charge of all project production. Others appointed to head various departments at the Institute include Dr. George W. Ward, inorganic chemistry; Dr. C. L. Shrewsbury, agricultural research; Dr. Frank H. Trimble, applied physics research; Dr. Carl M. Marberg, organic chemistry. The Institute staff now totals forty persons.

New General Electric Research Laboratory

A new building for the General Electric Company's Research Laboratory, which will afford some 50 percent more space than present facilities provide, will be erected near Schenectady at a cost of $8,000,000. Construction will begin as soon as WPB approval can be obtained.

The site has been a private estate known as "The Knolls," and includes 219 acres. It is in suburban Niskayuna, about 4½ miles from the main plant and offices in Schenectady. Overlooking the Mohawk River, it is on a rocky cliff which will afford an excellent and solid foundation for the laboratory buildings. The river at this point forms part of the New York State Barge Canal, and the Troy branch of the New York Central runs along the bank, so there will be ample water and rail transportation available. In addition the river will afford a good supply of water for laboratory use.

"The two buildings now occupied by the Laboratory were built in 1914 and 1922," said Dr. C. G. Suits, G-E Vice President and Director of Research, "and while they were the last word in laboratory construction then, this is no longer true. For some time we have been cramped for space and this condition has been aggravated in the past few years when all our facilities have been devoted to war work. We have a very much expanded program for the postwar years, which will increase our staff from its present 540 to about 800.

"We must look ahead even 15 or 20 years beyond this, when still further expansion may be required. We shall probably need to build smaller structures for special purposes, for example, a pilot plant for a new chemical process. All this requires not only a building that is immediately suitable, but one to which we can add and with ample grounds around it for the other buildings.

"Lack of space adjacent to the Schenectady Works, as a result of wartime construction which has utilized all prospective building sites, has forced a consideration of other locations. The one selected, in Niskayuna, and the laboratory building that we are planning, will meet these needs ideally. Every new and modern feature will be incorporated and I feel confident that we shall have one of the finest and most complete research institutions in the world."

The new building, in the general shape of the letter T, will vary from two to five stories in height and will include 200,000 square feet of laboratory working space in addition to an auditorium seating 300, a dining room, conference rooms, etc. One third of the laboratory space will be devoted to service facilities, machine shops, and specialty shops, such as glass blowers, all in a convenient central location.

Walls between rooms will be movable, capable of being placed at 18-inch intervals so that rooms may easily be made large or small as desired. Benches and all furnishings will be standardized so that they can easily be shifted from place to place as the need arises. The building will be air-conditioned throughout. Wires and pipes carrying various kinds and voltages of electricity, compressed air, suction, illuminating gas, hydrogen, oxygen, etc., will interlace the building whence they can be brought into any room.

The high elevation of the site above the river will afford many advantages, for example, in experiments with radar, high frequency jet engines. The rocky cliff foundation will be useful in conducting experiments with x-rays. These are being produced at 100 million volts in the present laboratory, and further increases are expected.

A base for flying laboratories designed to speed through scientific and mechanical tests new military and civilian aerial developments, such as the actual harnessing of jet engine and gas turbine power for new-type planes, will also be established by General Electric Company, at Schenectady County Airport. Work on a large hangar and ground laboratory will begin immediately at the airport. This hangar and laboratory will be 160 feet wide, 180 feet deep, and 45 feet high in the center. Two floors of laboratories, work rooms and offices will run down one side. In front will be a high control tower with all of the latest electronic and mechanical equipment.

McGill University Microscope

McGill University of Montreal, Canada, has installed an RCA electron microscope for extended work in physics, biology, medicine, chemistry, and metallurgy. Scientists at McGill are using the instrument for war research, but important peacetime studies are projected.